The Long Way from QCD gauge sector to pion's and nucleon's GPDs

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From gauge sector to hadron phenomenology

D. Binosi, L. Chang, J. Papavassiliou, C.D. Roberts,
Phys. Lett. B, 183 (2015)

Quark propagator gap equation:

$$S^{-1}(p) = Z_2\,(i\gamma\cdot p + m^{\rm bm}) + \Sigma(p)\,,$$

$$\Sigma(p) = Z_1 \int_{dq}^{\Lambda} g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S(q)\frac{\lambda^a}{2}\Gamma_\nu(q,p),$$

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Interaction kernel:

$$Z_1 g^2 D_{\mu\nu}(k)\Gamma_\nu(q,p) = k^2\mathcal{G}(k^2)D_{\mu\nu}^{\rm free}(k)\,Z_2\,\Gamma_\nu^A(q,p)$$
$$= \left[k^2\mathcal{G}_{\rm IR}(k^2) + 4\pi\tilde{\alpha}_{\rm pQCD}(k^2)\right] D_{\mu\nu}^{\rm free}(k)\,Z_2\,\Gamma_\nu^A(q,p),$$

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$$\mathcal{I}(k^2) = k^2\mathcal{G}(k^2)\,,$$

$$\mathcal{G}(k^2) = \frac{8\pi^2}{\omega^4} D\, e^{-k^2/\omega^2} + \frac{8\pi^2\gamma_m\,\mathcal{E}(k^2)}{\ln[\tau + (1 + k^2/\Lambda^2_{\rm QCD})^2]},$$

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Phys. Lett. B, 183 (2015)

Current models of the interaction kernel yields ground- and excited-states hadron masses within a 10-15% of accuracy compared to experiments!!!
C.D. Roberts et al.
Few Body Sys. 51, 1 (2011)

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More details in Binosi's Friday talk!!!

IR gauge-sector interaction kernel:

$$\mathcal{I}_{\hat{d}}(k^2) := k^2 \hat{d}(k^2) = \frac{\alpha_s(\zeta^2)\Delta(k^2;\zeta^2)}{[1+G^2(k^2;\zeta^2)]^2} = \left[\frac{1}{1-L(q^2)F(q^2)}\right]^2 \alpha_T(q^2)\,;$$

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Top-down approach

bottom-up approach

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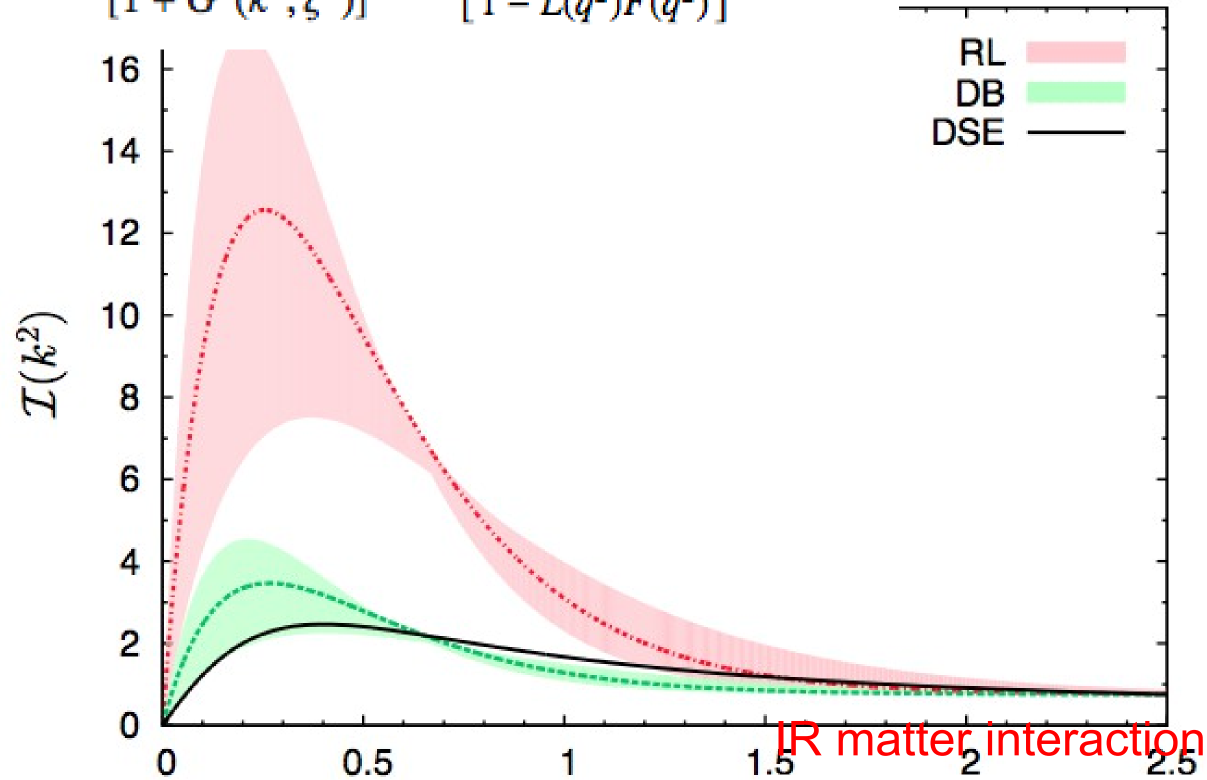

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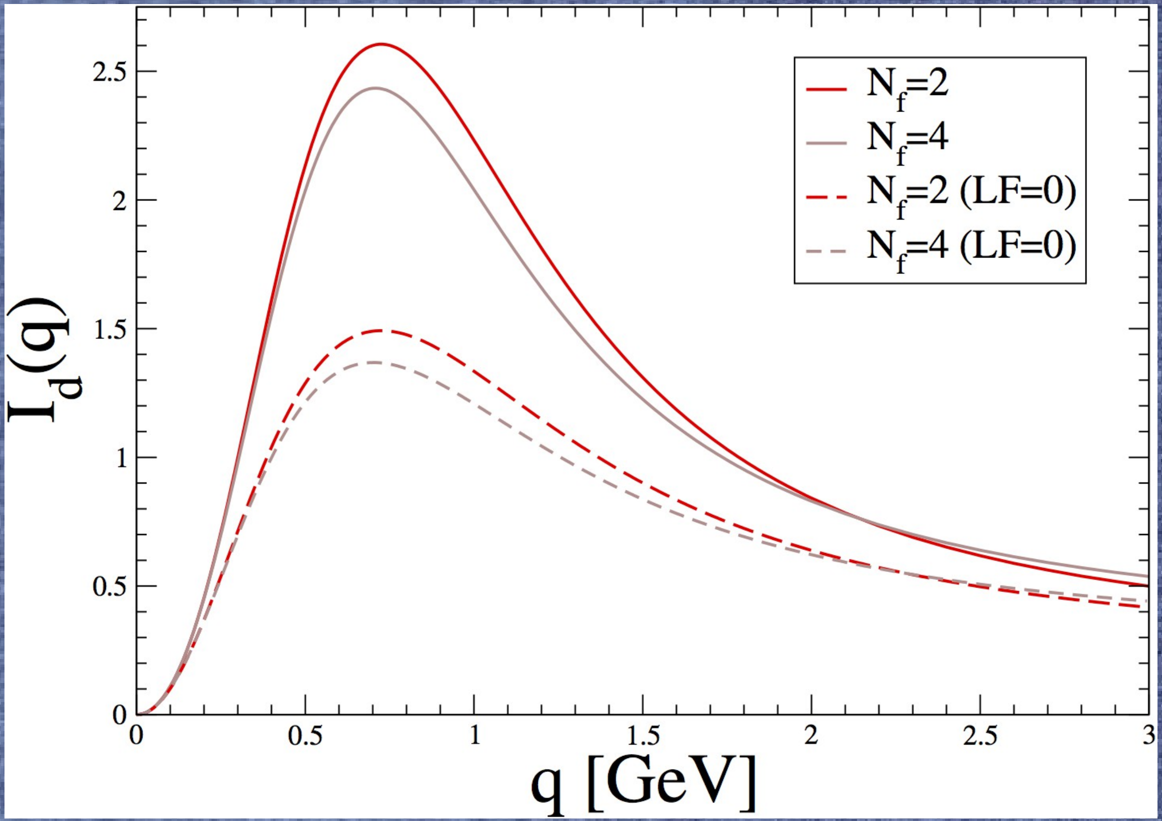
From gauge sector to hadron phenomenology

Very preliminary results!!!



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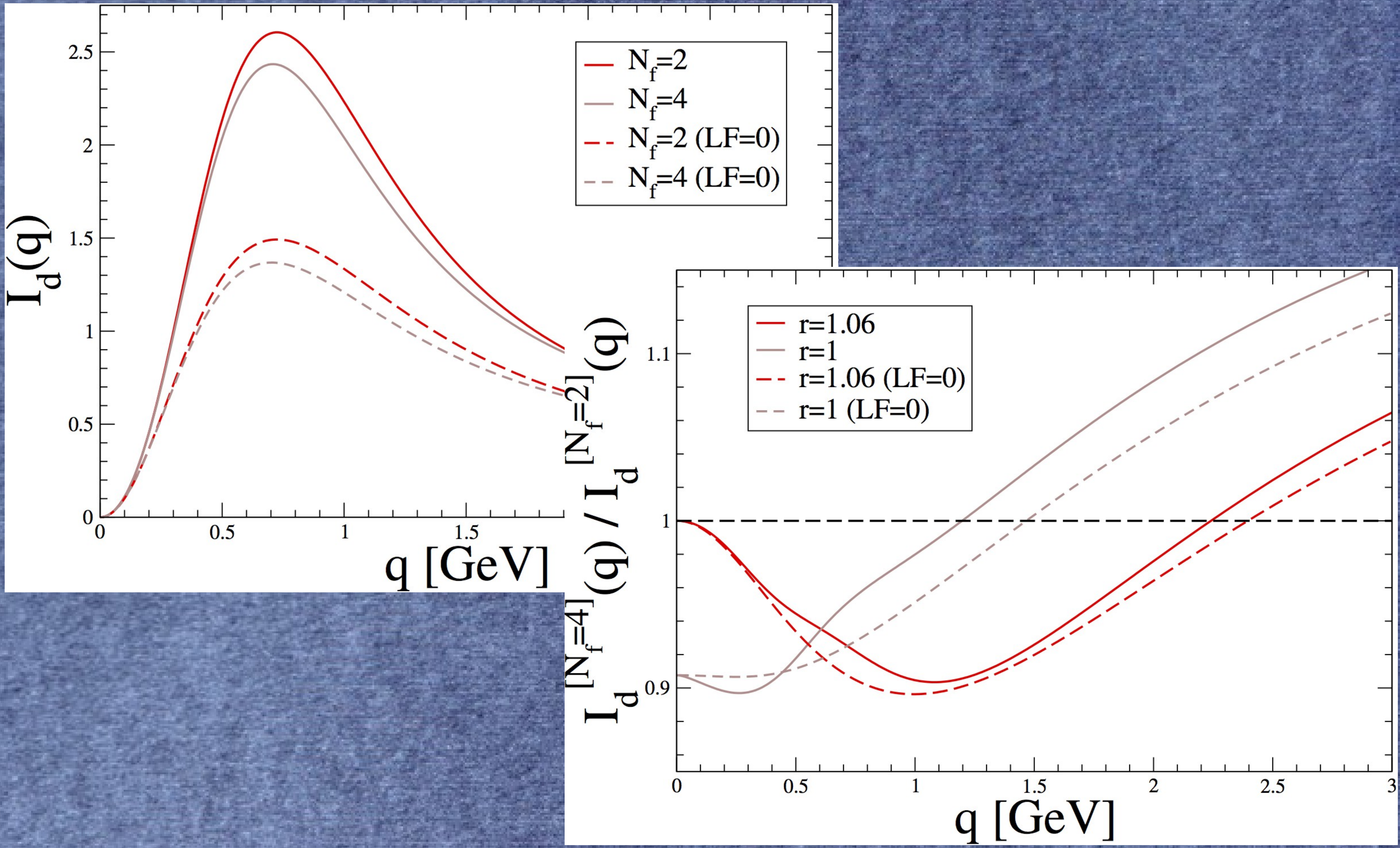

Why GPDs?

3D imaging of the nucleon's partonic content but also...

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- **Probabilistic interpretation** of Fourier transform of $GPD(x, \xi = 0, t)$ in **transverse plane**.

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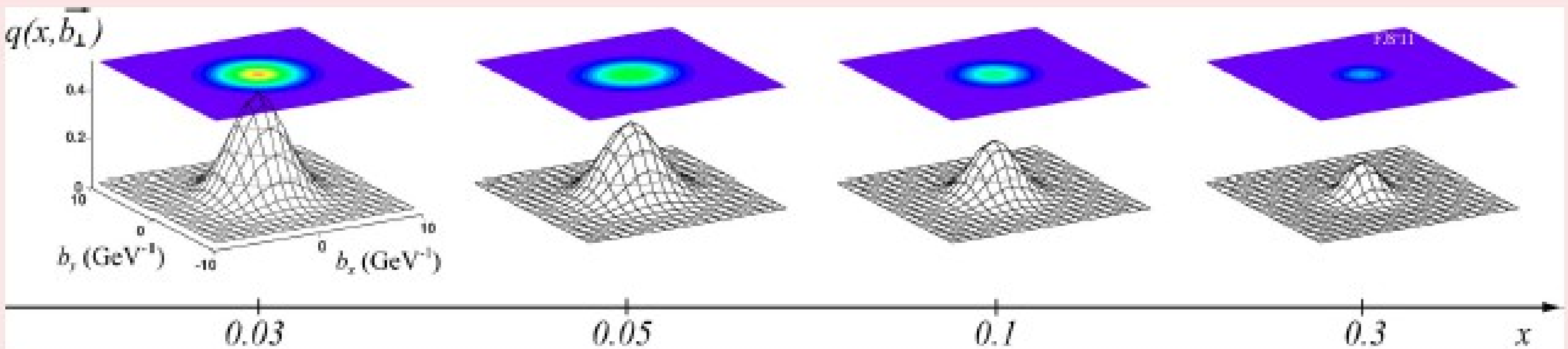
Transverse plane density (Goloskokov and Kroll model)



Why GPDs?

3D imaging of the nucleon's partonic content but also...

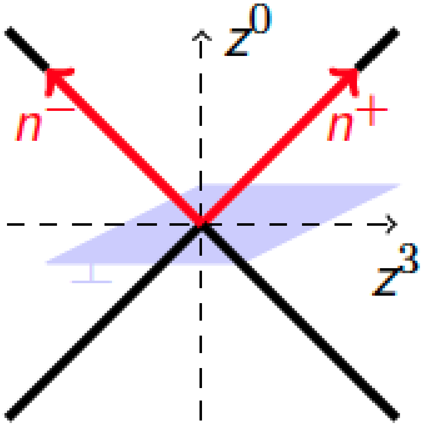
- Important topic for several **past, existing and future** experiments: H1, ZEUS, HERMES, CLAS, CLAS12, JLab Hall A, COMPASS, EIC, ...
- GPD modeling / parameterizing is an essential ingredient for the interpretation of experimental data.
- **Recent applications** of the Dyson-Schwinger and Bethe-Salpeter framework to **hadron structure studies**.

Pion GPD

Definition and symmetry properties:

$$H^q_\pi(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \middle| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t = \Delta^2$ and $\xi = -\Delta^+/(2P^+)$.



References

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Ji, Phys. Rev. Lett. **78**, 610 (1997)
Radyushkin, Phys. Lett. **B380**, 417 (1996)

- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Pion GPD

Form factors and parton distribution functions:

- PDF **forward limit**

$$H^q(x,0,0) = q(x)$$

Pion GPD

Form factors and parton distribution functions:

- PDF **forward limit**
- Form factor **sum rule**

$$\int_{-1}^{+1} dx\, H^q(x, \xi, t) = F_1^q(t)$$

Pion GPD

Form factors and parton distribution functions:

- PDF **forward limit**
- Form factor **sum rule**
- **Polynomiality**

$$\int_{-1}^{+1} dx\, x^n H^q(x,\xi,t) = \text{polynomial in } \xi$$

Pion GPD

Form factors and parton distribution functions:

- PDF **forward limit**
- Form factor **sum rule**
- **Polynomiality**
- **Positivity**

$$H^q(x,\xi,t) \leq \sqrt{q\left(\frac{x+\xi}{1+\xi}\right) q\left(\frac{x-\xi}{1-\xi}\right)}$$

Pion GPD

Form factors and parton distribution functions:

- PDF **forward limit**
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- H^q is an **even function** of ξ from time-reversal invariance.

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- H^q has support $x \in [-1, +1]$.
- **Soft pion theorem** (pion target)

$$H^q(x, \xi = 1, t = 0) = \frac{1}{2}\phi_\pi^q\left(\frac{1+x}{2}\right)$$

Pion GPD

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Numerous theoretical constraints on GPDs.

- There is no known GPD parameterization **relying only on first principles**.
- Modeling becomes a key issue.

Pion GPD

Double distributions (DD), a natural parameterization for a covariant GPD:

- A function satisfying a polynomiality property is the **Radon transform** of another function.
- Representation of GPD in terms of **Double Distributions**:

$$H^q(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi)\big(F^q(\beta,\alpha,t) + \xi G^q(\beta,\alpha,t)\big)$$

Müller *et al.*, Fortschr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Rev. **D59**, 014030 (1999)
Radysuhkin, Phys. Lett. **B449**, 81 (1999)

- Support property: $x \in [-1,+1]$.
- Discrete symmetries: F^q is α-even and G^q is α-odd.

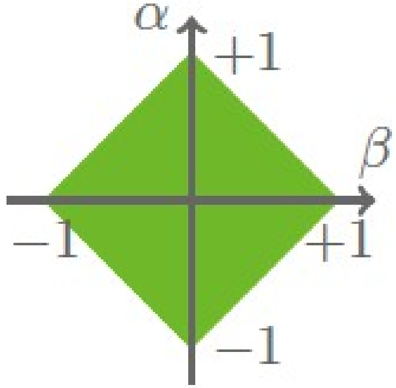
Pion GPD

Double distributions (DD), a natural parameterization for a covariant GPD:

- Define Double Distributions F^q and G^q as matrix elements of **twist-2 quark operators**:

$$\left\langle P+\frac{\Delta}{2}\middle|\bar{q}(0)\gamma^{\{\mu}i\overleftrightarrow{\mathrm{D}}^{\mu_1}\dots i\overleftrightarrow{\mathrm{D}}^{\mu_m\}}q(0)\middle|P-\frac{\Delta}{2}\right\rangle = \sum_{k=0}^{m}\binom{m}{k}$$

$$\left[F^q_{mk}(t)2P^{\{\mu}-G^q_{mk}(t)\Delta^{\{\mu}\right]P^{\mu_1}\dots P^{\mu_{m-k}}\left(-\frac{\Delta}{2}\right)^{\mu_{m-k+1}}\dots\left(-\frac{\Delta}{2}\right)^{\mu_m\}}$$



with

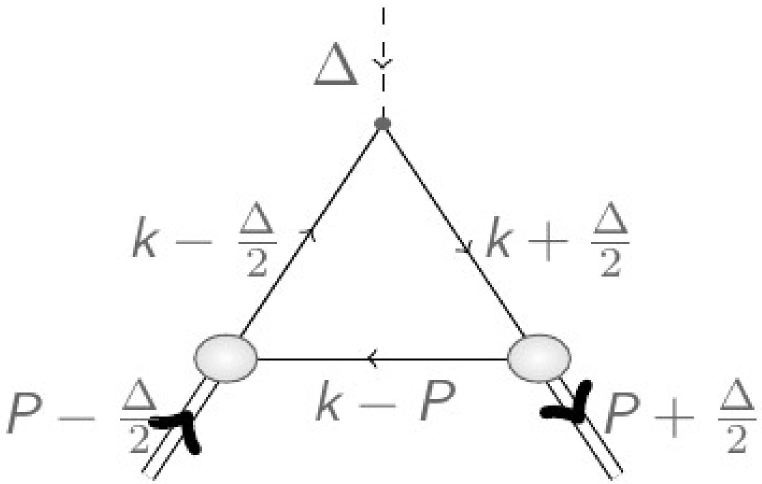
$$F^q_{mk} = \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}F^q(\beta,\alpha)$$

$$G^q_{mk} = \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\,\alpha^k\beta^{m-k}G^q(\beta,\alpha)$$

GPD in the DSE-BSE approach

Evaluation *via* the triangle diagram approximation:

$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD *H*.

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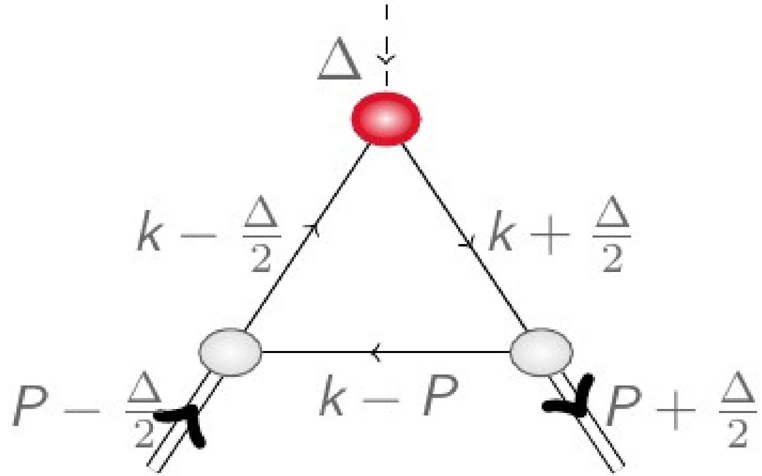

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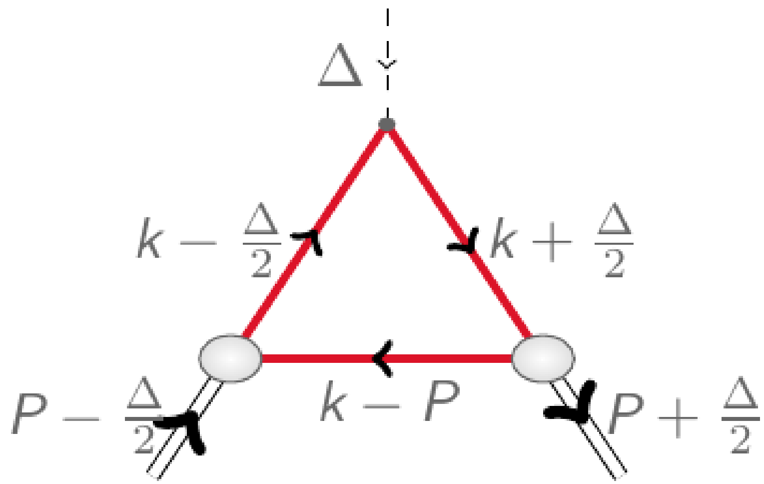
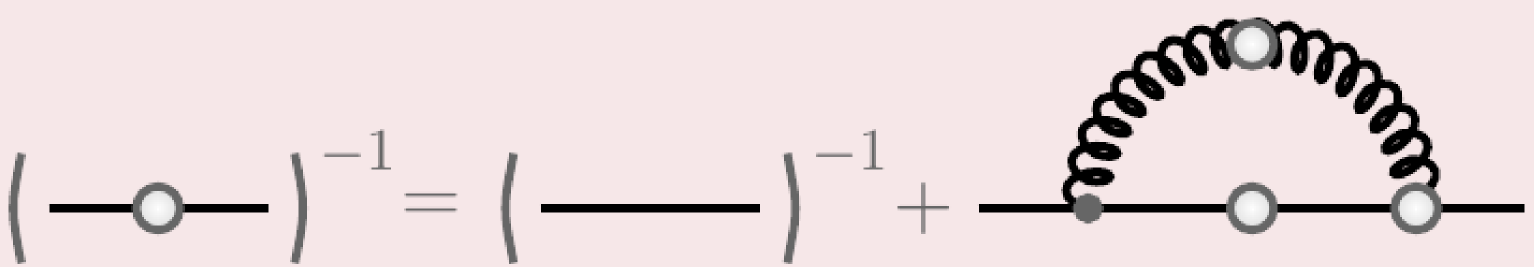


- Compute **Mellin moments** of the pion GPD *H*.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Dyson - Schwinger equation



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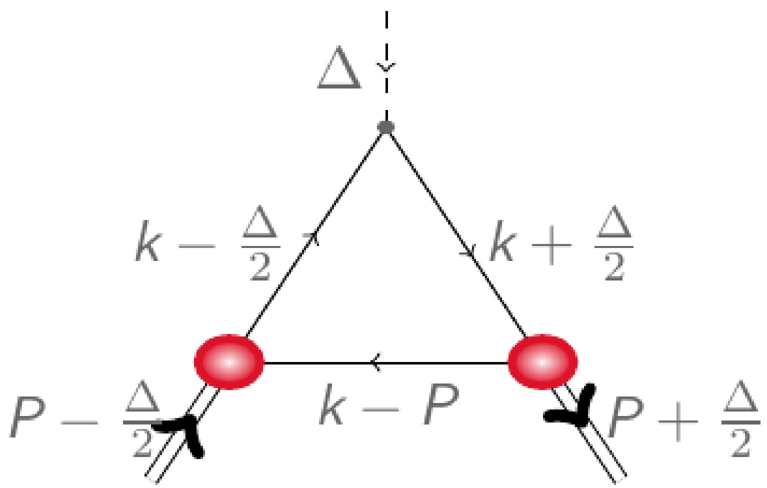

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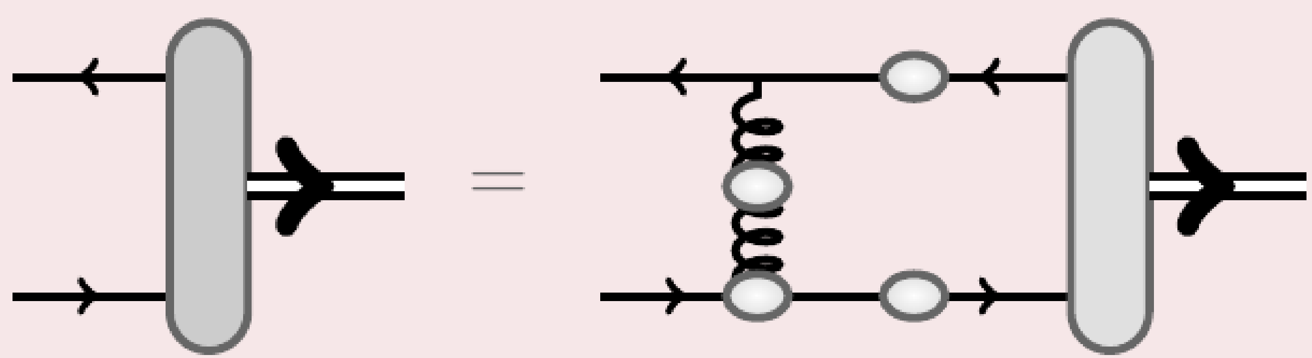
Bethe - Salpeter equation



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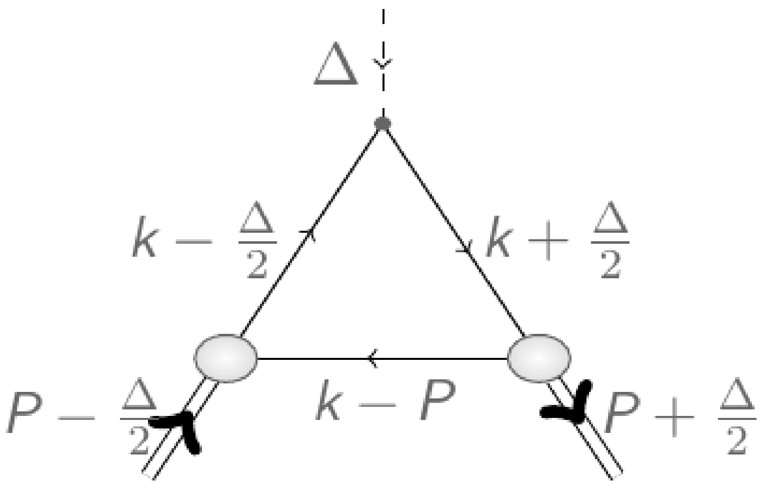

- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.
- **Nonperturbative** modeling.

GPD in the DSE-BSE approach

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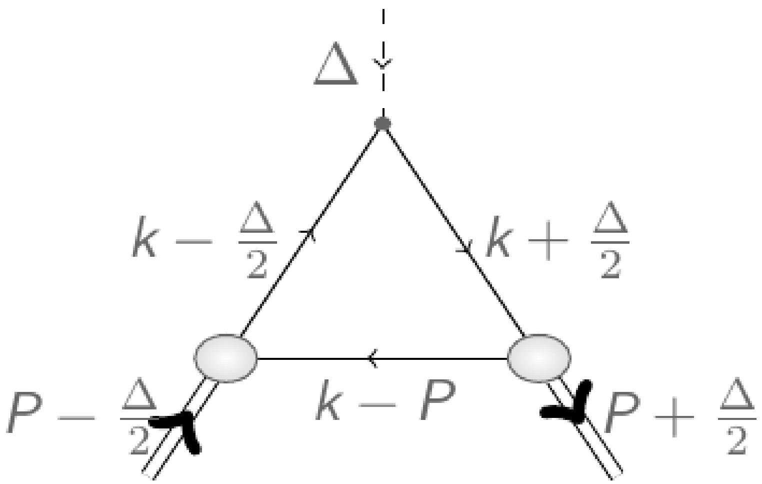

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- Most GPD properties **satisfied by construction**.

GPD in the DSE-BSE approach

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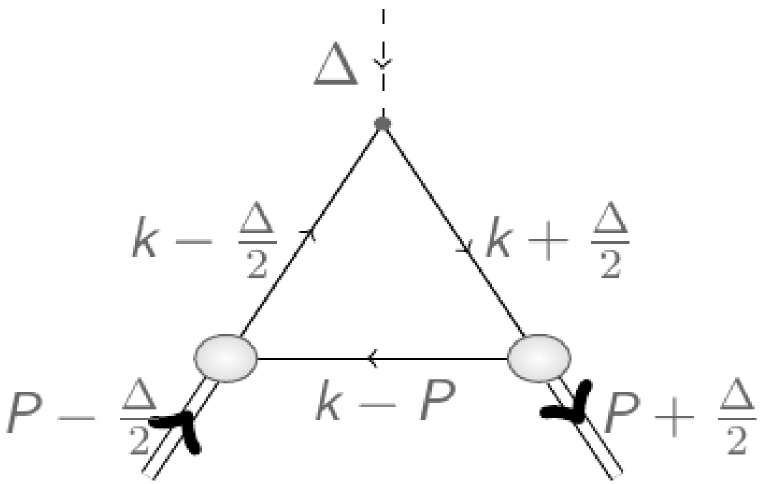


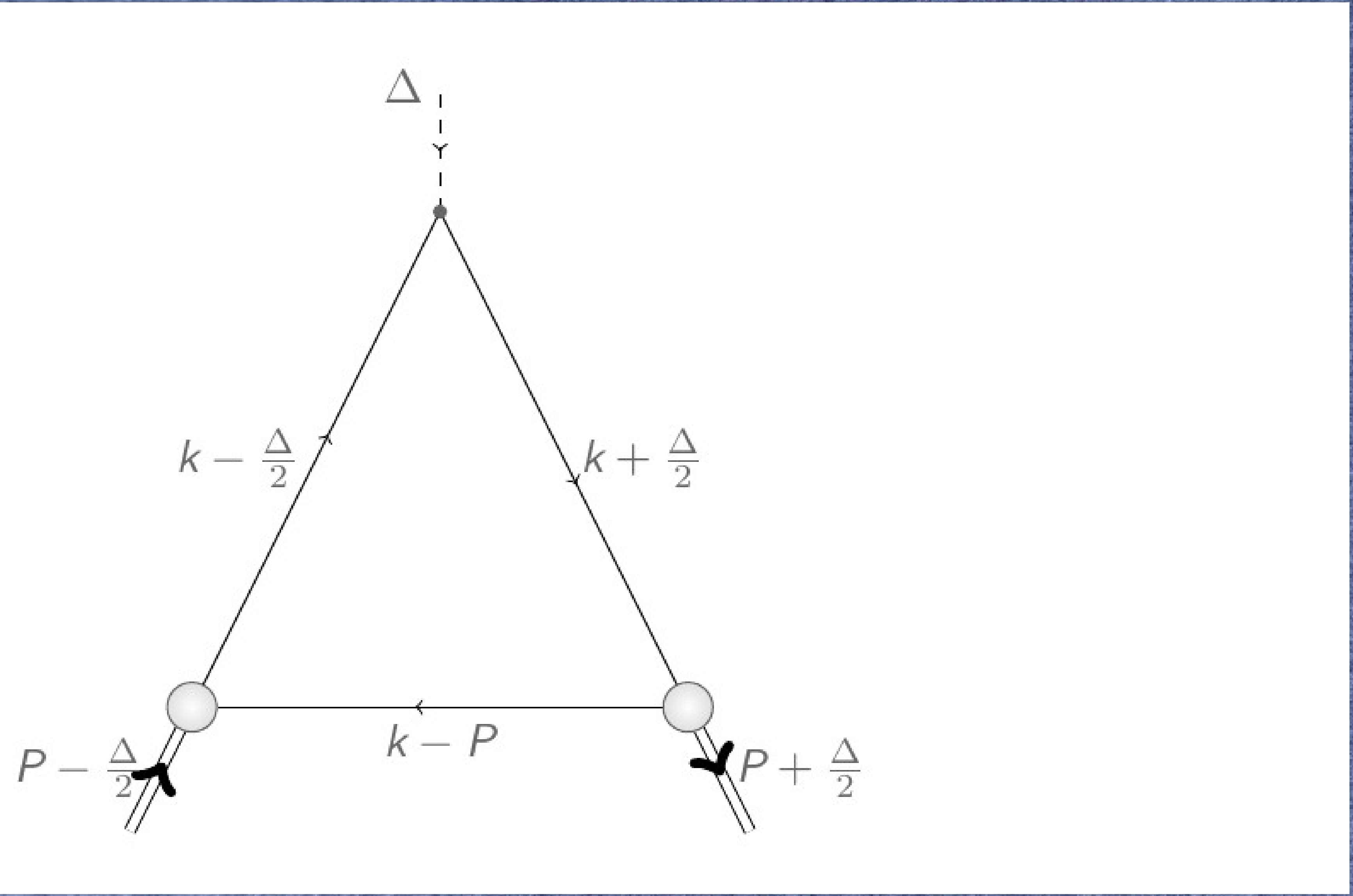
- Compute **Mellin moments** of the pion GPD *H*.
- Triangle diagram approx.
- Resum **infinitely many** contributions.
- **Nonperturbative** modeling.

- Most GPD properties **satisfied by construction**.
- Also compute crossed triangle diagram.

Mezrag *et al.*, arXiv:1406.7425 [hep-ph]
and Phys. Lett. **B741**, 190 (2015)

GPD in the DSE-BSE approach

Rainbow-ladder and physical content:



GPD in the DSE-BSE approach

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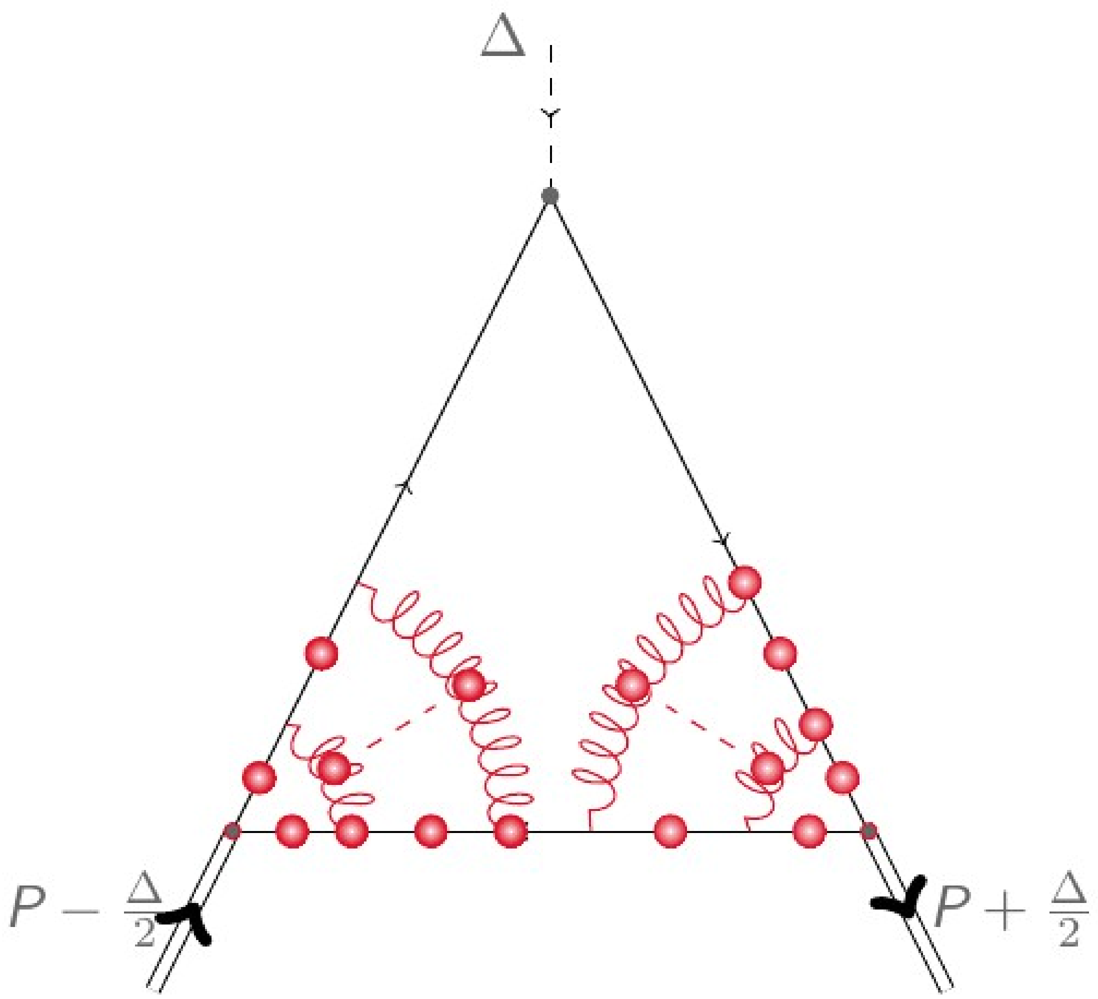

- Bethe-Salpeter vertex.

GPD in the DSE-BSE approach

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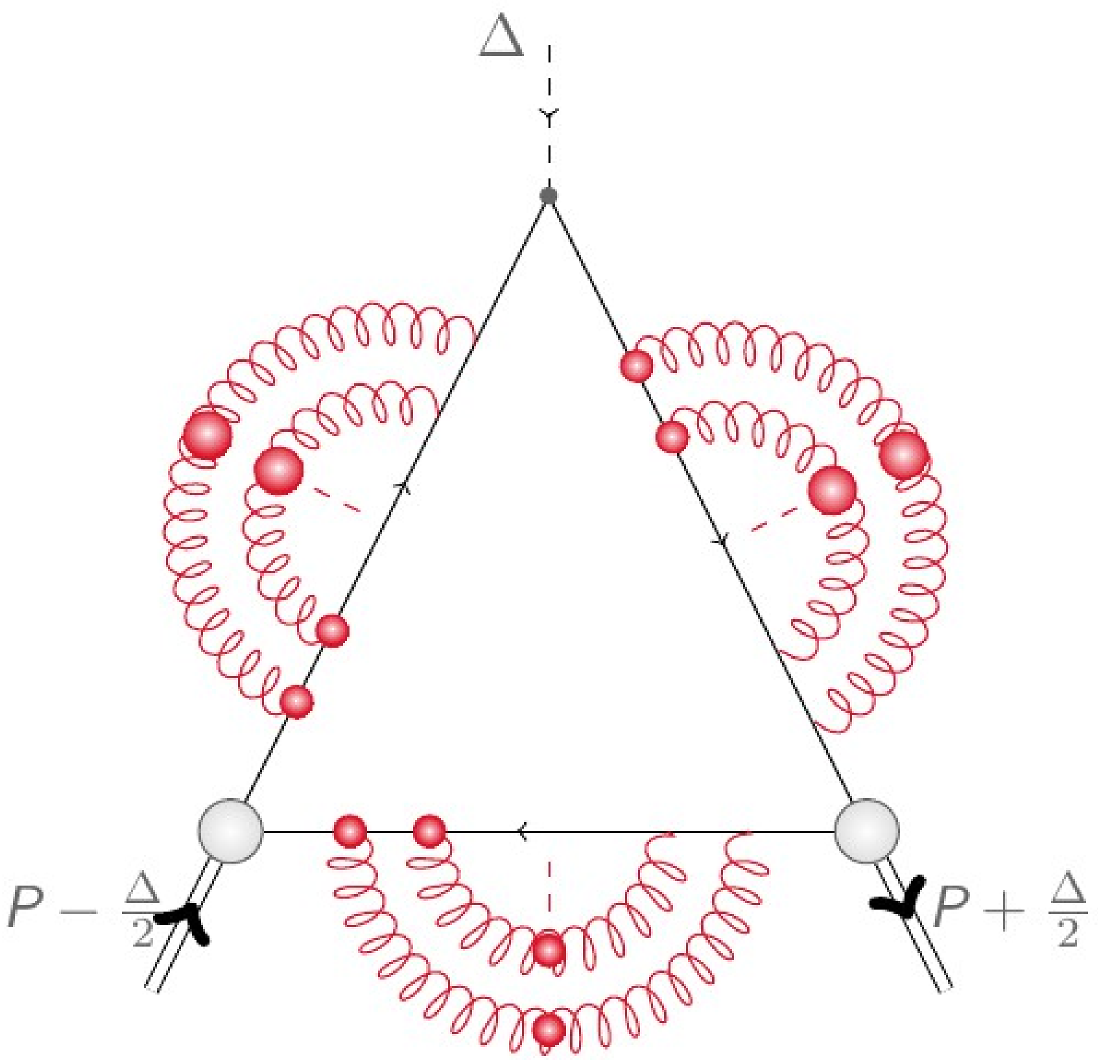

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- Dressed quark propagator.

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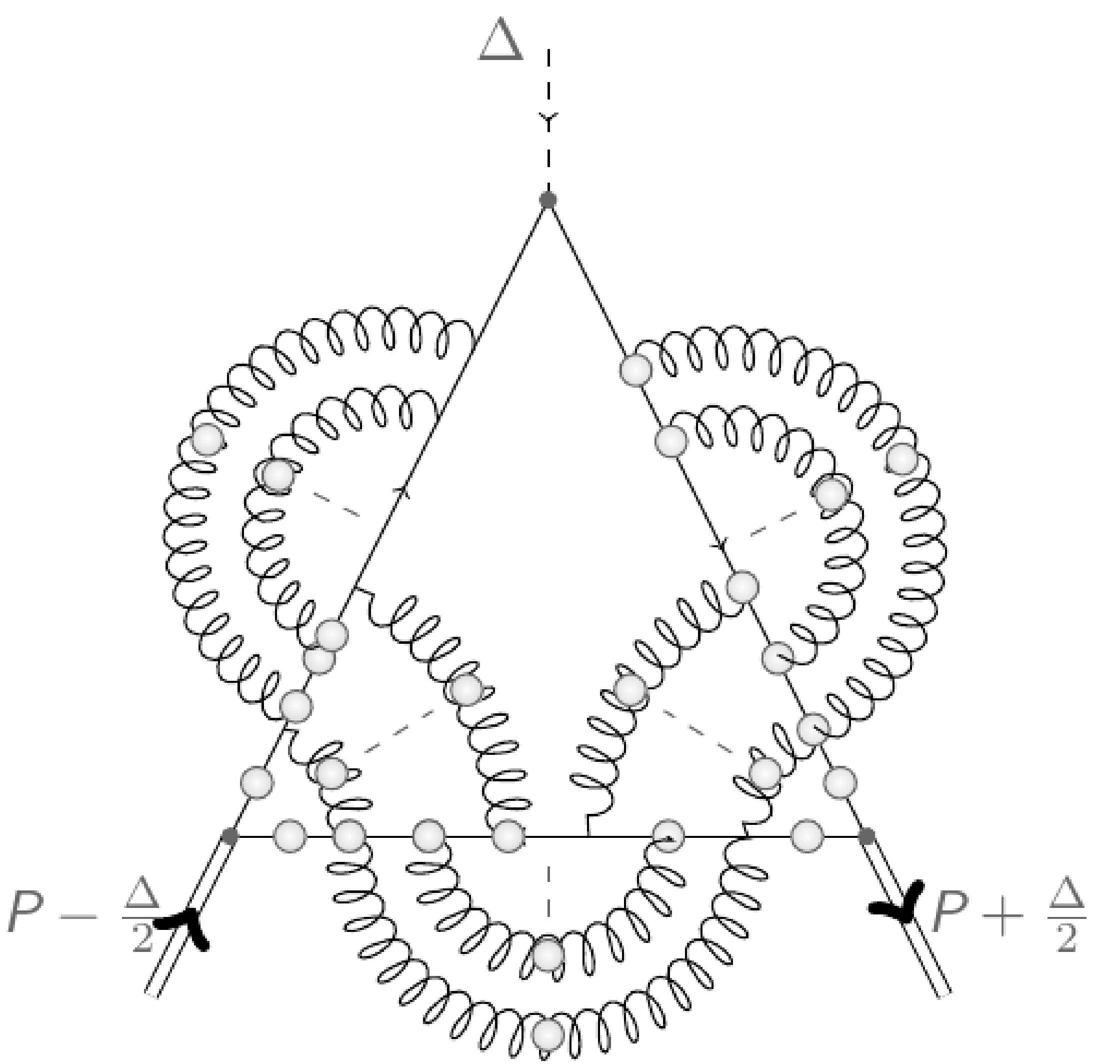

- Bethe-Salpeter vertex.
- Dressed quark propagator.
- Much more than tree level perturbative diagram!
- Enable description of **non perturbative phenomena**.

GPD in the DSE-BSE approach

Most of the properties made sure by construction:

- **Polynomiality** from Poincaré covariance.

GPD in the DSE-BSE approach

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- **Soft pion theorem** from **symmetry-preserving** truncation of Bethe-Salpeter and gap equations.

Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)

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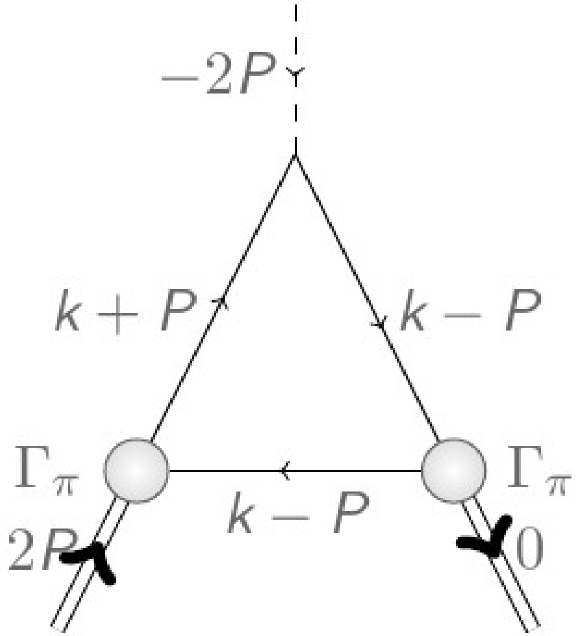

- Mellin moments.
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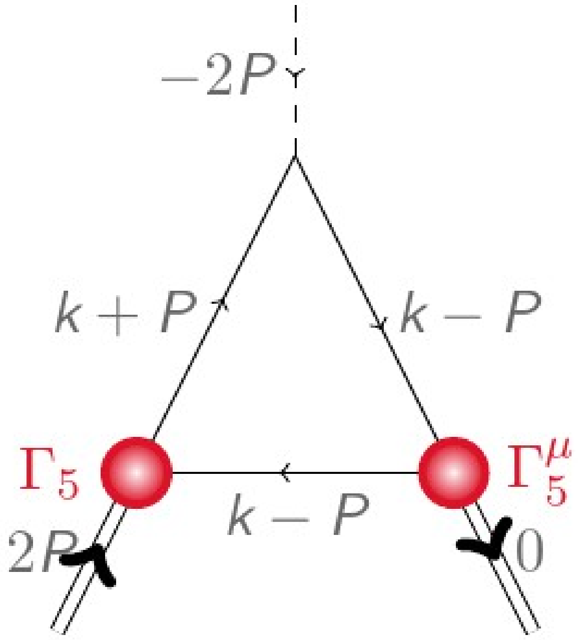

- Mellin moments.
- Soft pion kinematics.
- Axial and axial vector vertices Γ_5, Γ_5^μ in chiral limit.

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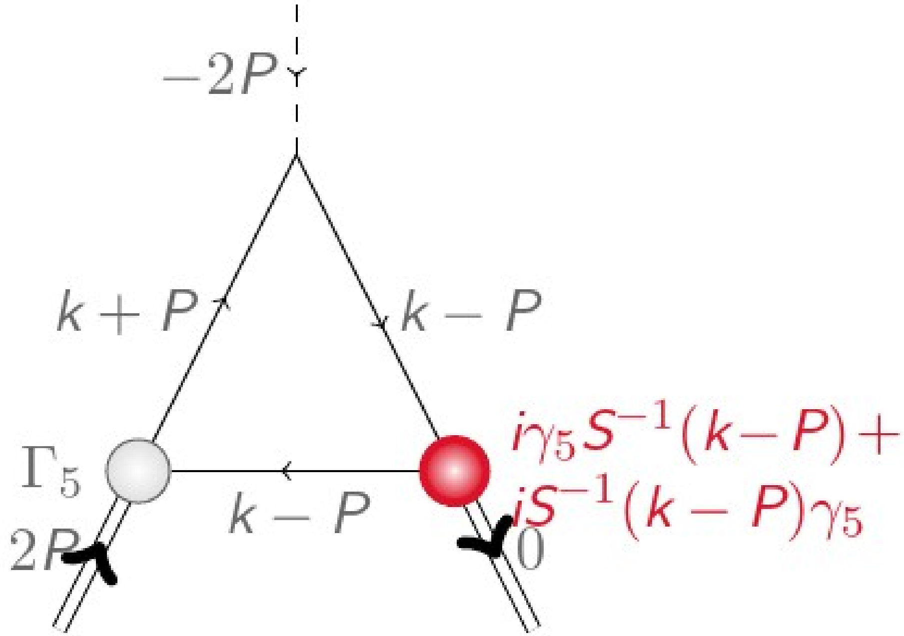
Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)



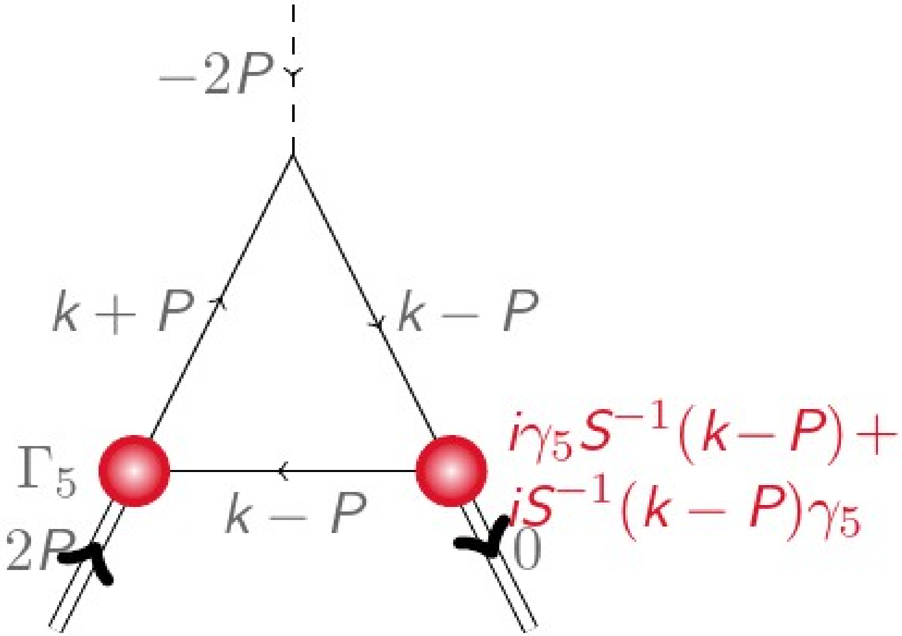
- Mellin moments.
- Soft pion kinematics.
- Axial and axial vector vertices Γ_5, Γ_5^μ in chiral limit.
- Axial-vector Ward identity.

GPD in the DSE-BSE approach

Most of the properties made sure by construction:

- **Polynomiality** from Poincaré covariance.
- **Soft pion theorem** from **symmetry-preserving** truncation of Bethe-Salpeter and gap equations.

Mezrag *et al.*, Phys. Lett. **B741**, 190 (2015)



- Mellin moments.
- Soft pion kinematics.
- Axial and axial vector vertices Γ_5, Γ_5^μ in chiral limit.
- Axial-vector Ward identity.
- Recover pion DA Mellin moments.

Algebraic DSE-BSE inspired GPD model

Have to deal with DSEs and BSEs solutions:

- Numerical resolution of gap and Bethe-Salpeter equations in Euclidean space.
- Analytic continuation to Minkowskian space required.
- **Ill-posed** problem in the sense of Hadamard.
- Parameterize solutions and fit to numerical solution:

Gap Complex-conjugate pole representation:

$$S(k) = \sum_{i=0}^{N} \left[\frac{z_i}{i\not{k} + m_i} + \frac{z_i^*}{i\not{k} + m_i^*} \right]$$

Bethe-Salpeter Nakanishi representation of amplitude $\mathcal{F}_\pi$:

$$\mathcal{F}_\pi(q^2, q \cdot P) = \int_{-1}^{+1} \mathrm{d}\alpha \int_0^\infty \mathrm{d}\lambda \frac{\rho(\alpha, \lambda)}{(q^2 + \alpha q \cdot P + \lambda^2)^n}$$

Algebraic DSE-BSE inspired GPD model

A first intermediate step before dealing with numerical solutions:

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[- i\gamma \cdot p + M \right] \Delta_M(p^2) \\
\Delta_M(s) &= \frac{1}{s + M^2} \\
\Gamma_\pi(k, p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z \, \rho_\nu(z) \, \left[\Delta_M(k_{+z}^2) \right]^\nu \\
\rho_\nu(z) &= R_\nu (1 - z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1 - z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:

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Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Results for the pion GPD

Verification of the theoretical constraints:

- **Analytic expression** in the DGLAP region.

$$H^u_{x\geq\xi}(x,\xi,0) = \frac{48}{5}\left\{ \frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3} \right.$$

$$\frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3}$$

$$+\frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}$$

$$+\frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3}$$

$$+\frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3}$$

$$+\frac{3\left(\left(15(2x(x+5)+5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right)}{20\left(\xi^2-1\right)^3}$$

$$\left. +\frac{3\left(2x(5x(x+2)-6)+15\xi^6-5\xi^2+3\right)\log\left(1-\xi^2\right)}{20\left(\xi^2-1\right)^3}\right\}$$

Results for the pion GPD

Verification of the theoretical constraints:

- **Analytic expression** in the DGLAP region.
- Similar expression in the ERBL region.

Results for the pion GPD

Verification of the theoretical constraints:

- **Analytic expression** in the DGLAP region.
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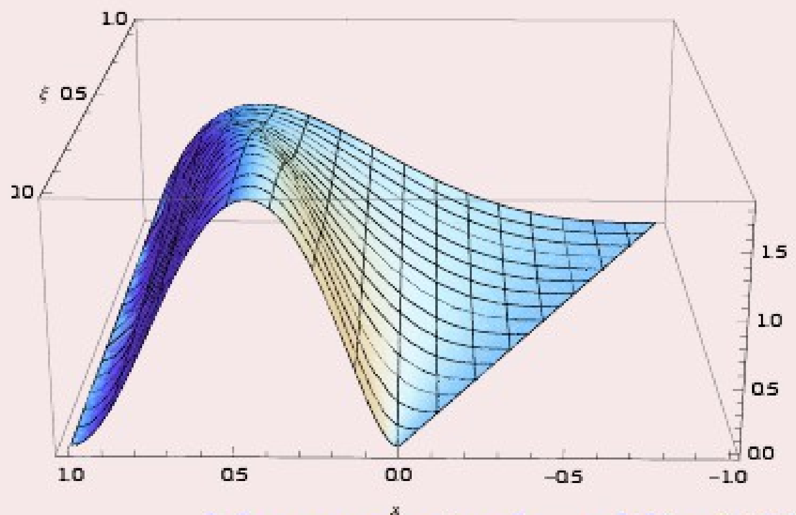
- Also direct verification using Mellin moments of H.

Valence $H^u(x, \xi, t)$ as a function of x and ξ at vanishing t.



Mezrag *et al.*, arXiv:1406.7425 [hep-ph]

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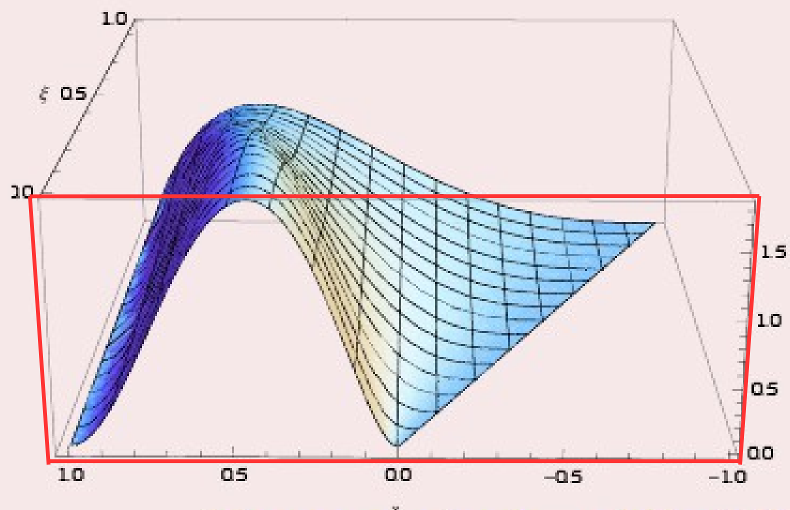

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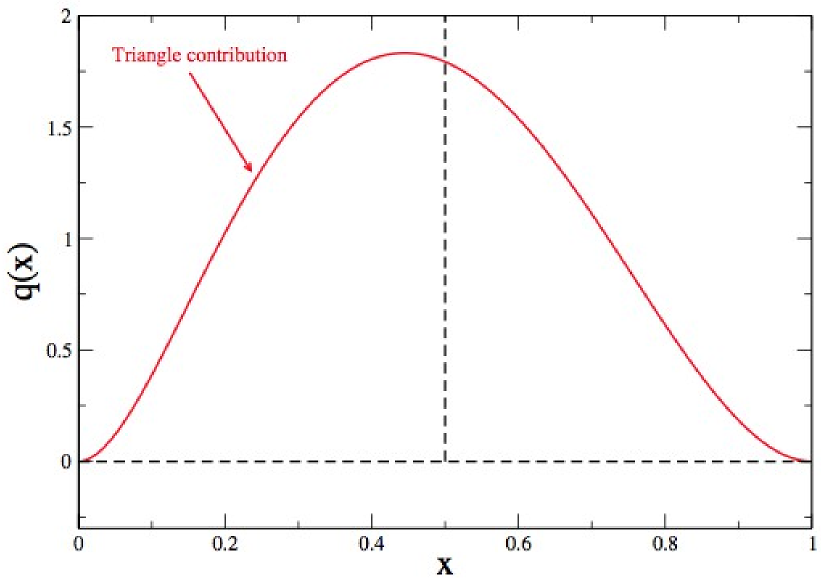
Results for the pion GPD

The two-body problem:



$$q_A^{\pi}(x) = n_q \Big[x^3(x[-2(x-4)x-15]+30)\ln(x) + (2x^2+3) \times (x-1)^4 \ln(1-x) + x[x(x[2x-5]-15)-3](x-1) \Big] \, ,$$

- The PDF appears not to be symmetric around $x = \frac{1}{2}$.

Results for the pion GPD

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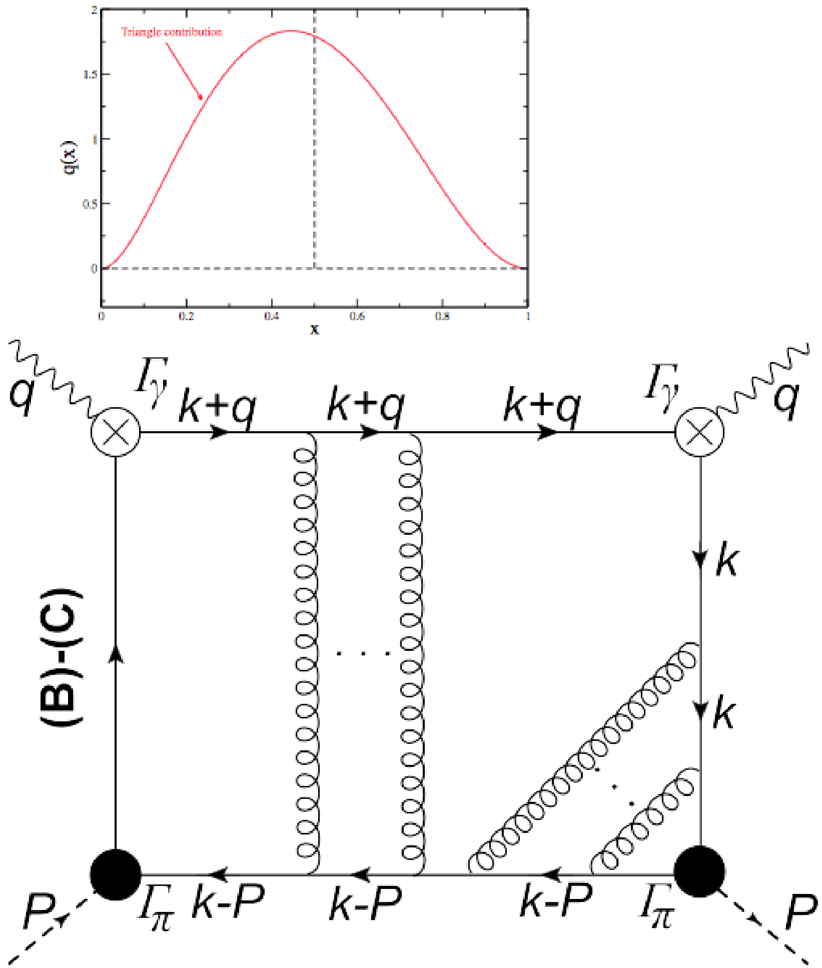

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Results for the pion GPD

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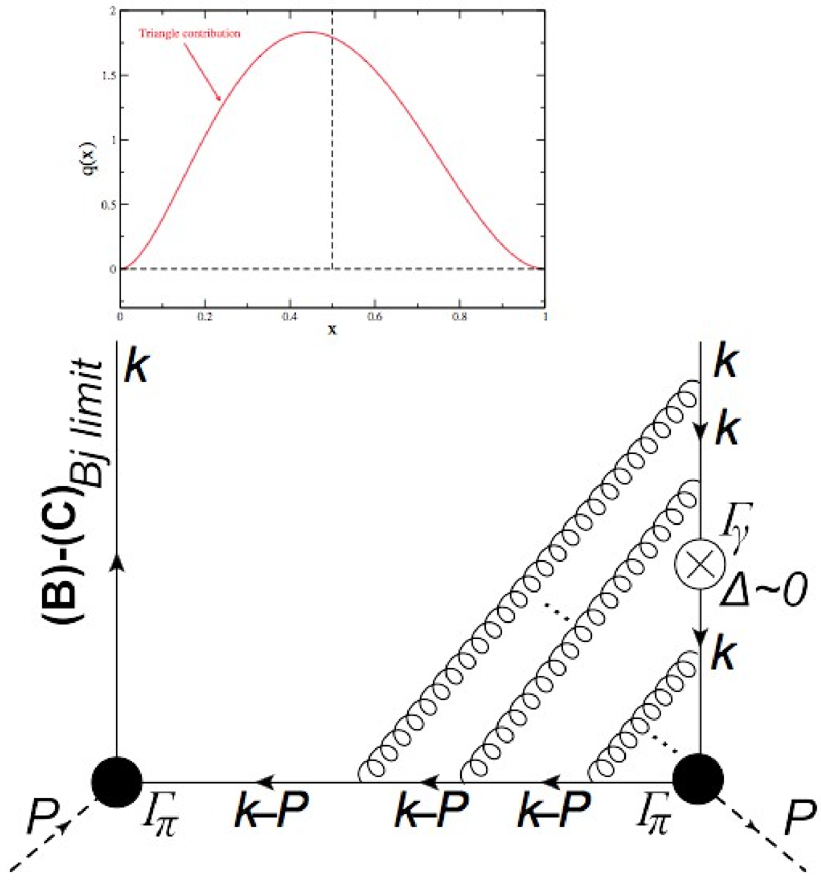

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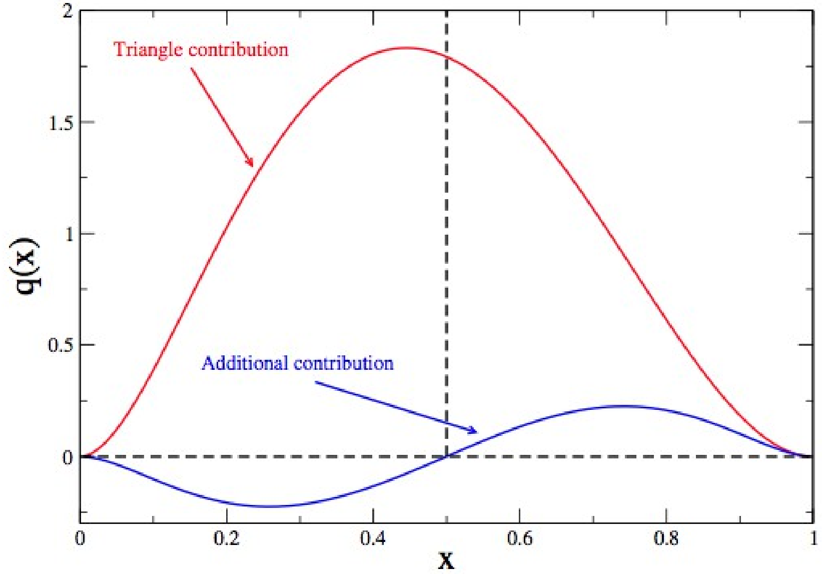

$$q_A^{\pi}(x) = n_q \Big[x^3(x[-2(x-4)x-15]+30)\ln(x) + (2x^2+3) \times (x-1)^4 \ln(1-x) + x[x(x[2x-5]-15)-3](x-1) \Big] ,$$

$$q_{BC}^{\pi}(x) = n_q \Big[x^3(2x([x-3]x+5)-15)\ln(x) - (2x^3+4x+9) \times (x-1)^3 \ln(1-x) - x(2x-1)([x-1]x-9)(x-1) \Big] . \quad (13)$$

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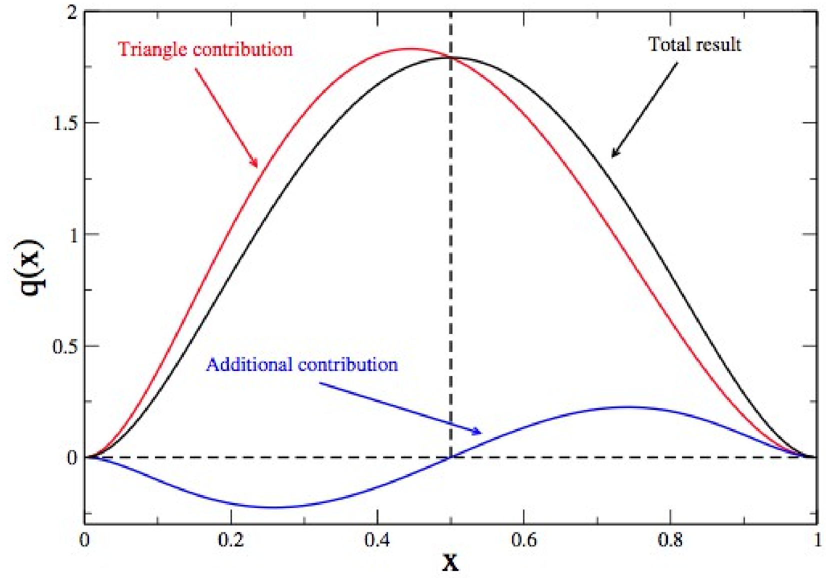
Results for the pion GPD

The two-body problem:



$$q_L^\pi(x) = \frac{72}{25}\left[x^3(x[2x-5]+15)\ln(x) + (x[2x+1]+12)\times(1-x)^3\ln(1-x) + 2x(6-[1-x]x)(1-x)\right].$$

- The PDF appears not to be symmetric around $x = \frac{1}{2}$.
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- Adding this contribution allows us to recover a symmetric PDF [L. Chang *et al.*, Phys.Lett.B737(2014)2329].

Results for the pion GPD

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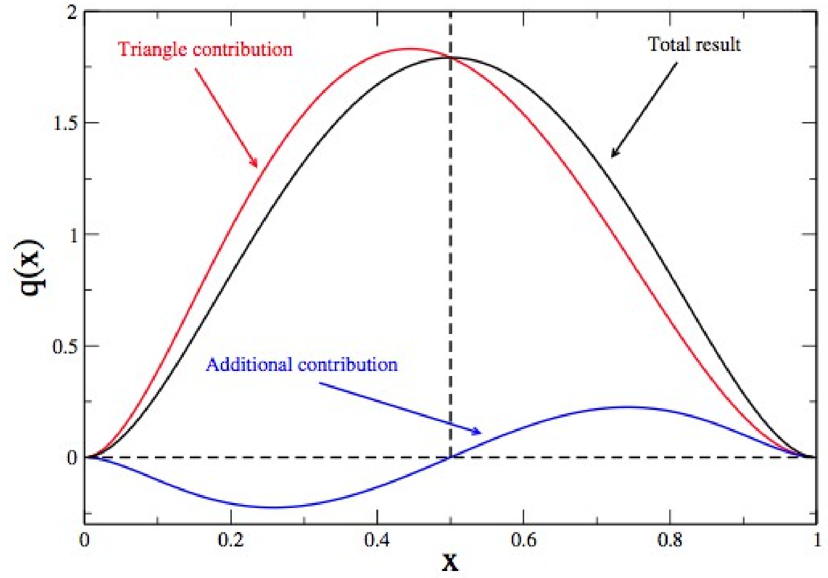

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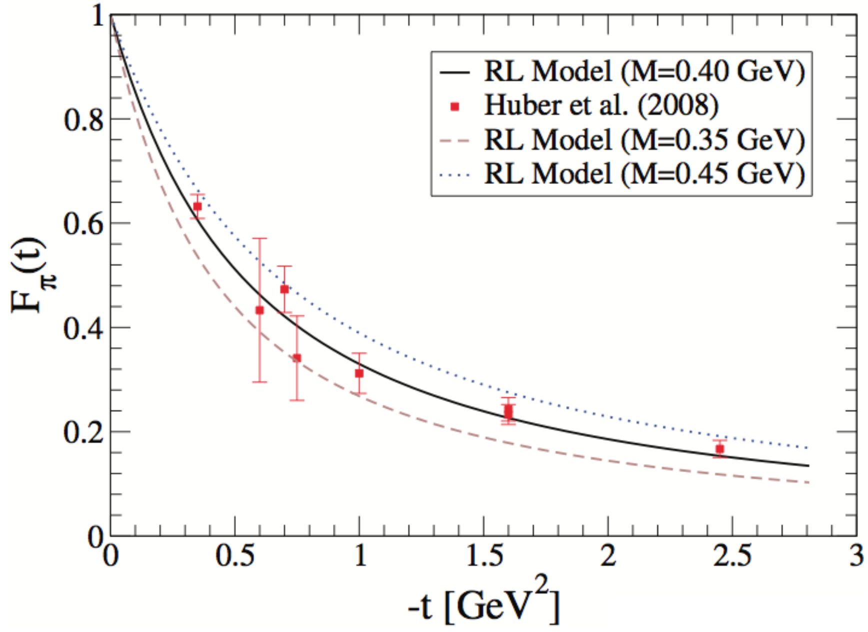
Results for the pion GPD

The form factor and the dimensionful parameter:

- Pion form factor obtained from isovector GPD:

$$\int_{-1}^{+1} \mathrm{d}x\, H^{I=1}(x,\xi,t) = 2F_{\pi}(t)$$

- Single dimensionful parameter $M \simeq 400$ MeV.



C. Mezrag et al., Phys. Lett. B190 (2015) 741

Results for the pion GPD

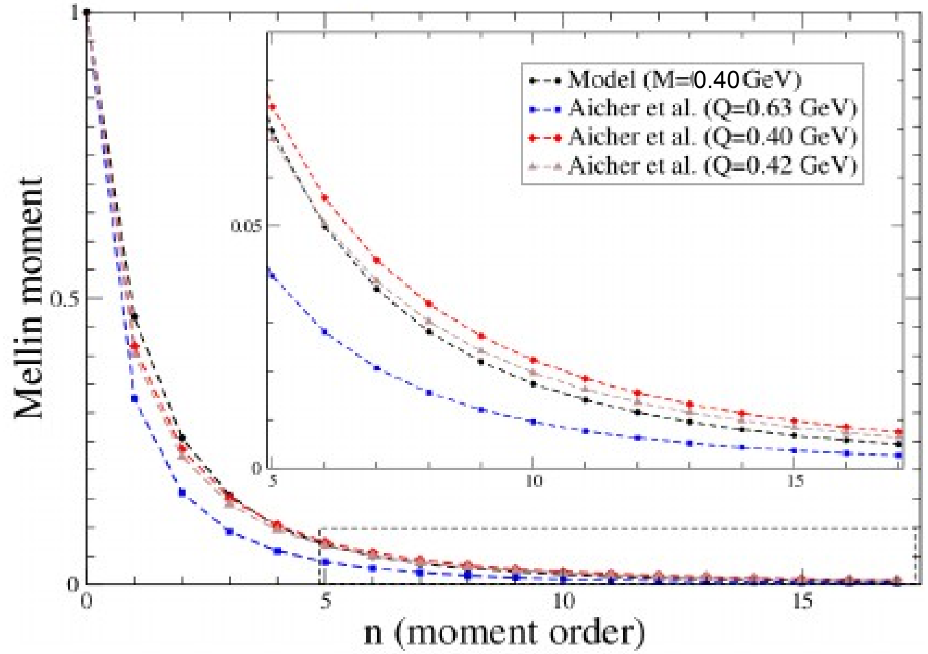
The parton distribution function:

- Pion PDF obtained from forward limit of GPD:

$$q(x) = H^q(x, 0, 0)$$

- Use LO DGLAP equation and compare to PDF extraction.

  Aicher *et al.*, Phys. Rev. Lett. **105**, 252003 (2010)

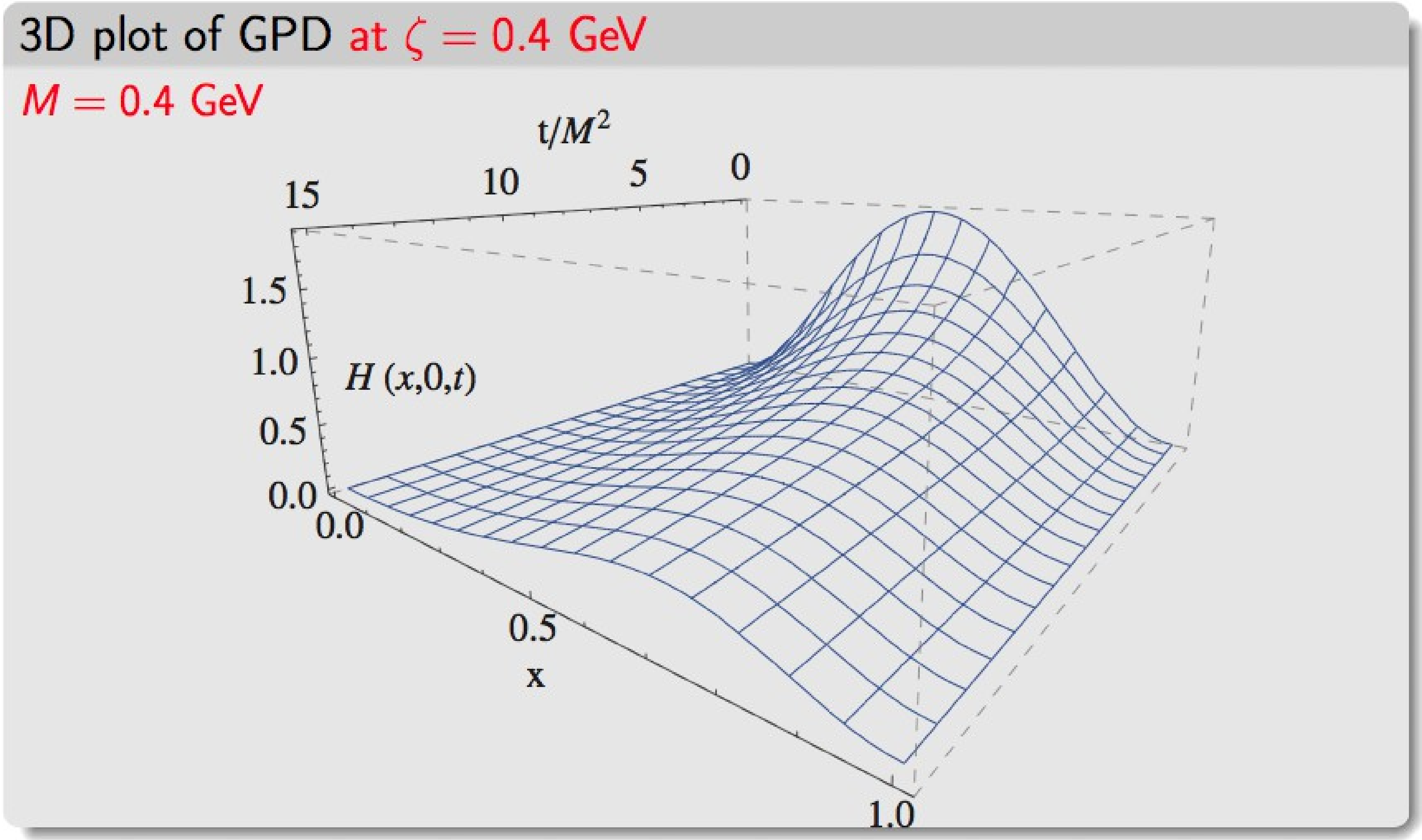


Mezrag *et al.*, arXiv:1406.7425 [hep-ph]

- Find model initial scale $\mu \simeq 400$ MeV.

Results for the pion GPD

The off-forward (non-skewed) GPD:



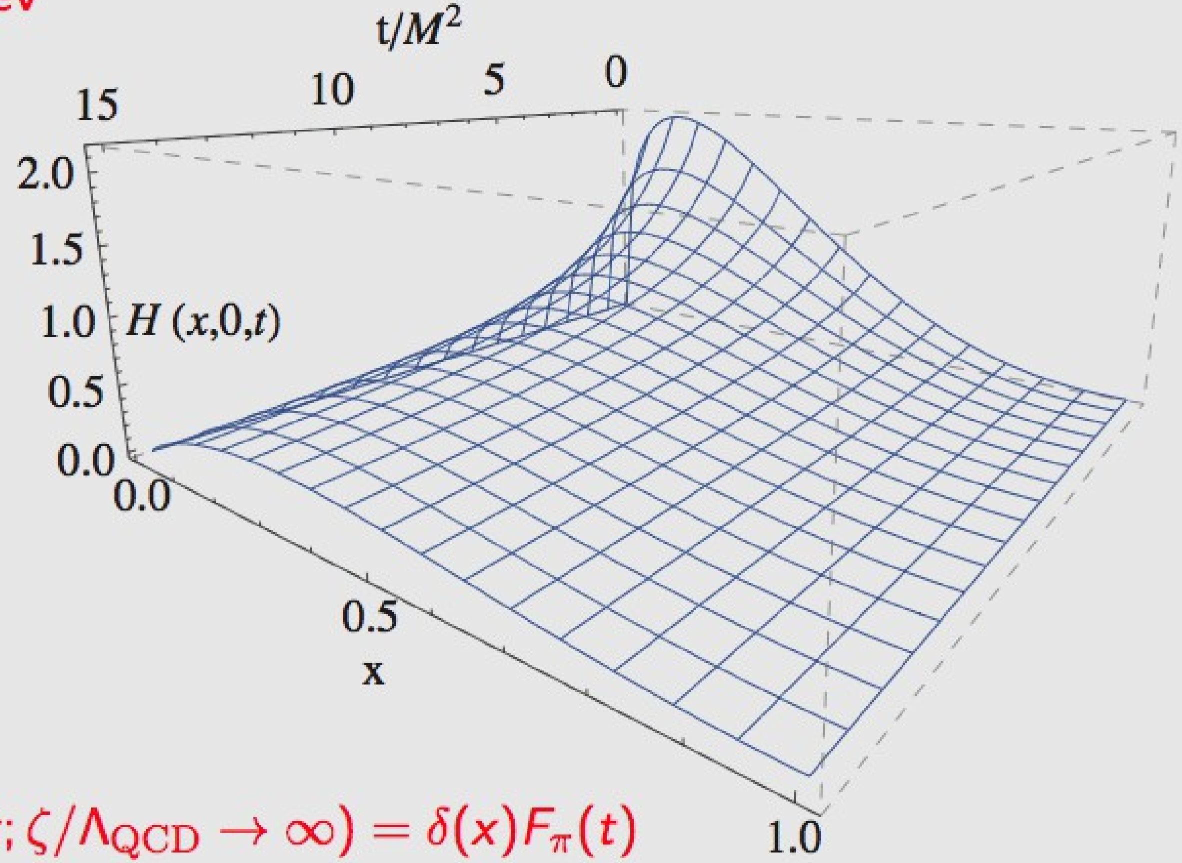
C. Mezrag et al., Phys. Lett. B190 (2015) 741

Results for the pion GPD

The off-forward (non-skewed) GPD:

3D plot of GPD at $\zeta = 2$ GeV (DGLAP running; $x > \xi$)

$M = 0.4$ GeV



$$H(x, 0, t; \zeta/\Lambda_{QCD} \to \infty) = \delta(x) F_\pi(t)$$

C. Mezrag et al., Phys. Lett. B190 (2015) 741

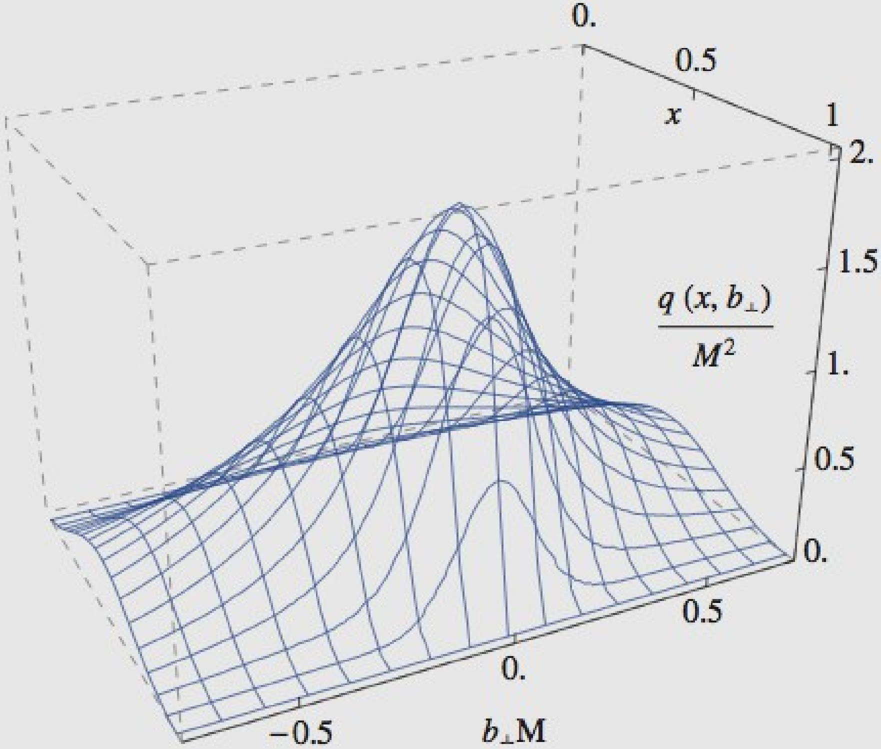
Results for the pion GPD

The off-forward (non-skewed) GPD:

$$q(x, |\vec{b}|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi} |\vec{\Delta}_\perp| J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|) H(x, 0, -\Delta_\perp^2)$$

Impact parameter space GPD at $\zeta = 0.4$ GeV

$M = 0.4$ GeV



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Results for the pion GPD

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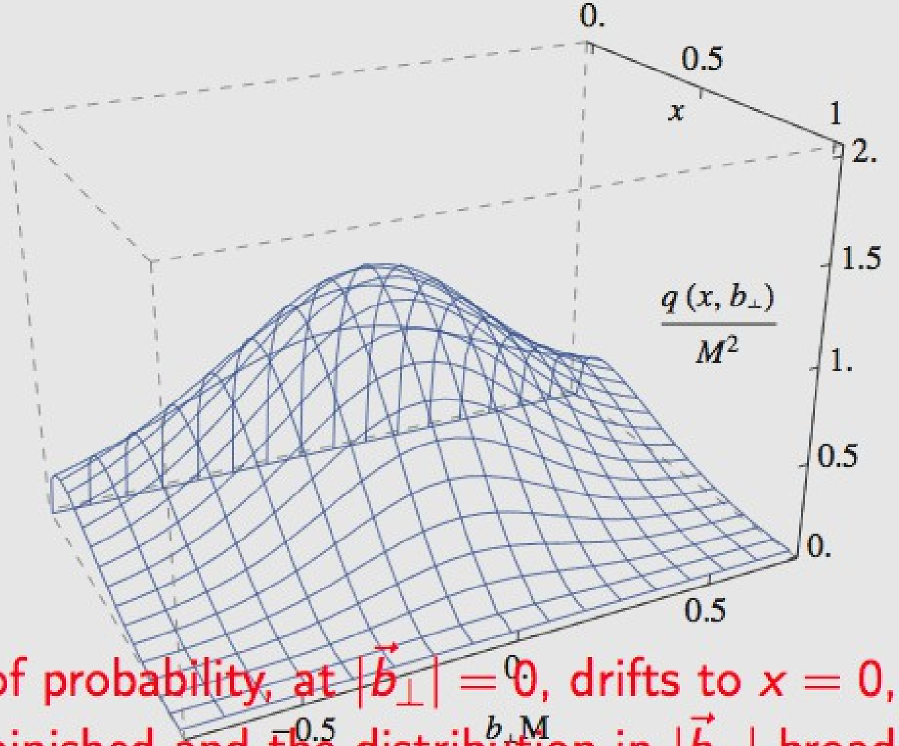


The peak of probability, at $|\vec{b}_\perp| = 0$, drifts to $x = 0$, its height is diminished and the distribution in $|\vec{b}_\perp|$ broadens.

C. Mezrag et al., Phys. Lett. B190 (2015) 741

Results for the pion GPD

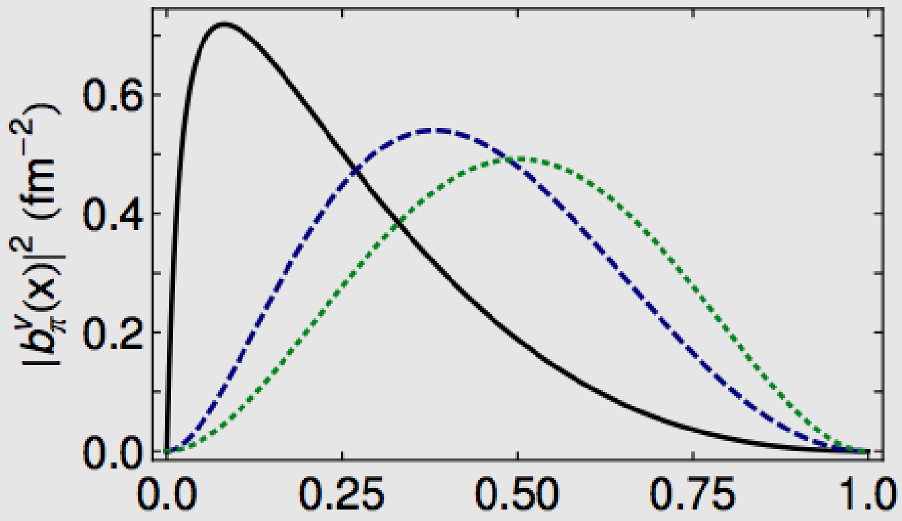
The off-forward (non-skewed) GPD:

$$q(x,|\vec{b}|) = \int \frac{d|\vec{\Delta}_\perp|}{2\pi}|\vec{\Delta}_\perp|J_0(|\vec{b}_\perp||\vec{\Delta}_\perp|)H(x,0,-\Delta_\perp^2)$$

$$\langle|\vec{b}_\perp|^2\rangle = \int_{-1}^{1} dx\, \langle|\vec{b}_\perp(x;\zeta)|^2\rangle = \int_{-1}^{1} dx \int_0^\infty d|\vec{b}_\perp||\vec{b}_\perp|^3 \int_0^\infty d\Delta\Delta J_0(\vec{b}_\perp|\Delta)F_\pi(\Delta^2)$$

Impact parameter space GPD

$r_\pi = \sqrt{3/2\langle|\vec{b}_\perp|^2\rangle} = 0.674$ fm $\Longleftrightarrow$ $r_\pi = 0.672(8)$ fm [PRD86(2012)010001]



$\zeta = 2$ GeV; $\zeta = 0.4$ GeV; $\zeta = 0.4$GeV [c(x,t)=1].

C. Mezrag et al., Phys. Lett. B190 (2015) 741

The overlap approach

A first-principle connection with Light-Front Wave Function:

- Decompose an hadronic state $|H; P, \lambda\rangle$ in a Fock basis:

$$|H; P, \lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x \mathrm{d}\mathbf{k}_\perp]_N \psi_N^{(\beta,\lambda)}(x_1, \mathbf{k}_{\perp 1}, \ldots, x_N, \mathbf{k}_{\perp N}) \, |\beta, k_1, \ldots, k_N\rangle$$

- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H^q(x, \xi, t) \propto \sum_{\beta,j} \int [\mathrm{d}\bar{x} \mathrm{d}\bar{\mathbf{k}}_\perp]_N \delta_{j,q} \delta(x - \bar{x}_j) \psi_N^{(\beta,\lambda)*}(\hat{x}', \hat{\mathbf{k}}'_\perp) \psi_N^{(\beta,\lambda)}(\tilde{x}, \tilde{\mathbf{k}}_\perp)$$

  with $\tilde{x}, \tilde{\mathbf{k}}_\perp$ (resp. $\hat{x}', \hat{\mathbf{k}}'_\perp$) generically denoting incoming (resp. outgoing) parton kinematics.

  Diehl *et al.*, Nucl. Phys. **B596**, 33 (2001)

- Similar expression in the ERBL region $-\xi \leq x \leq \xi$, but with overlap of N- and $(N+2)$-body LFWF.

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- Derive an expression for the pion GPD in the DGLAP region $\xi \leq x \leq 1$:

$$H_\pi^q(x,\xi,t)_{\xi\leq x\leq 1} = C^q \int d^2\mathbf{k}_\perp^2 \Psi^*\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}; P_-\right)\Psi\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}; P_+\right)$$

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- Evaluate LFWF in algebraic model:

$$\Psi(k^+, \mathbf{k}_\perp; P) = -\frac{1}{2\sqrt{3}} \int \frac{dk^-}{2\pi} \mathrm{Tr}\left[\gamma^+ \gamma_5 \chi_\pi(k, P)\right]$$

The overlap approach

A first-principle connection with Light-Front Wave Function:

- Evaluate LFWF in algebraic model:

$$\psi(x, \mathbf{k}_\perp) \propto \frac{x(1-x)}{[(\mathbf{k}_\perp - x\mathbf{P}_\perp)^2 + M^2]^2}$$

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- Expression for the GPD at $t = 0$:

$$H(x, \xi, 0) \propto \frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}$$

The overlap approach

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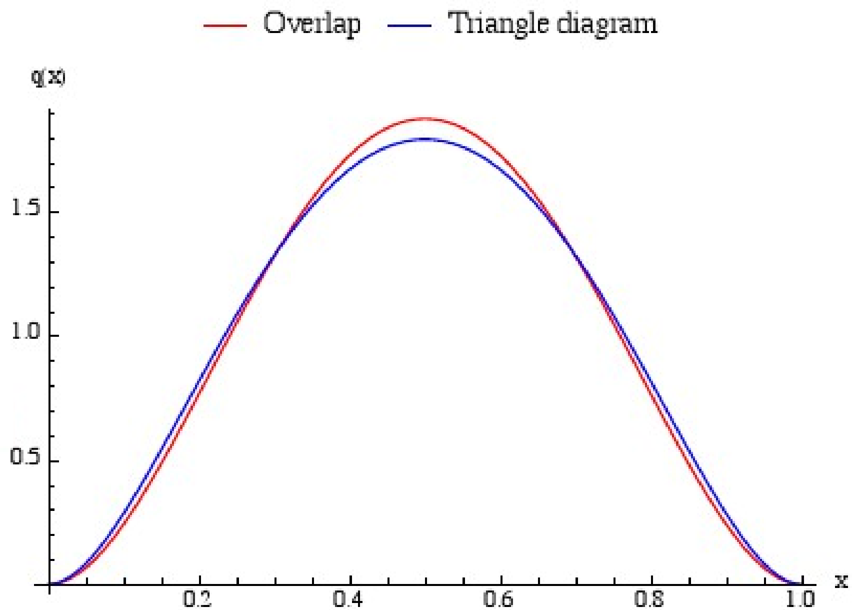

- Manifest 2-body symmetry.
- Expression for the PDF:

$$q(x) = 30x^2(1-x)^2$$

- Off-forward case: *in progress*.

DGLAP to ERBL extension from the overlap

Before dealing with a nucleon GPD, one needs to solve the problem of the extension to $\xi > x$:

1CDD-scheme Radon transform:

$$\frac{\sqrt{1-\xi^2}}{x} H(x,\xi,t) = \mathcal{R}f(s,\varphi,t) = \int_{|\alpha|+|\beta|\leq 1} d\beta\, d\alpha\, \delta(s - \beta\cos\varphi - \alpha\sin\varphi)\, f(\beta,\alpha,t)$$

$$\begin{aligned} x &= \frac{s}{\cos\phi} \\ \xi &= \tan\phi \end{aligned}$$

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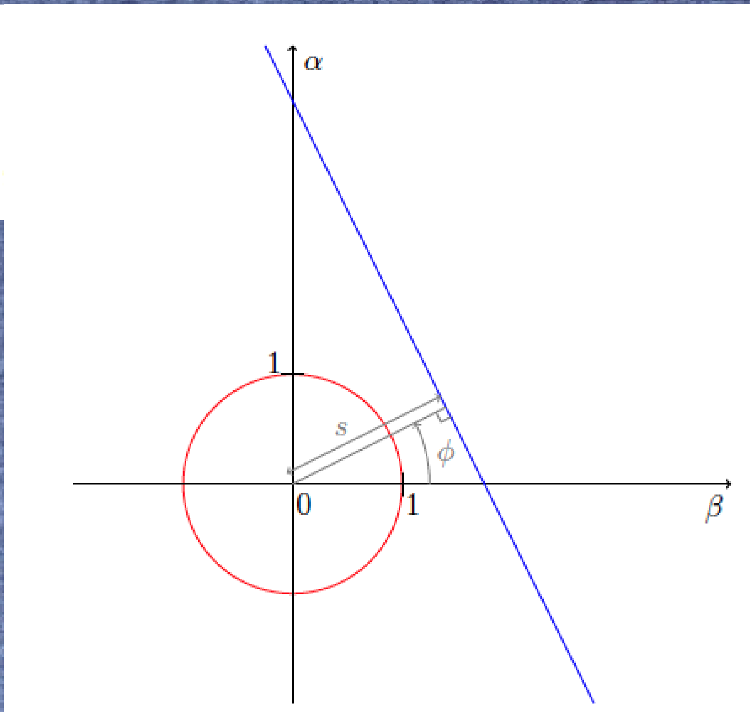

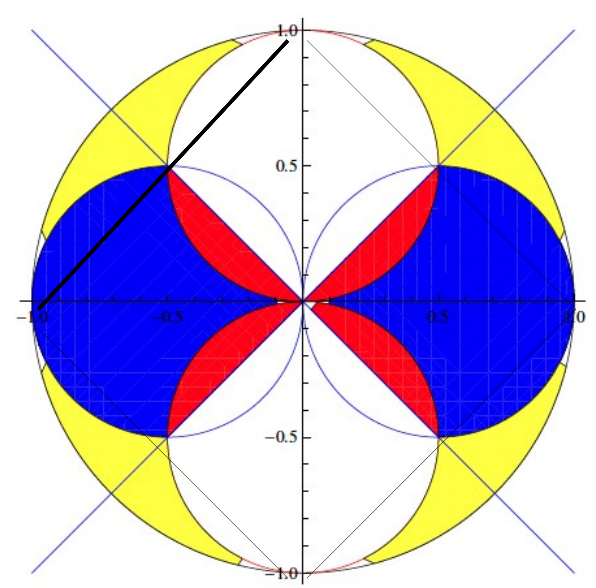

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$$\frac{\sqrt{1-\xi^2}}{x} H(x,\xi,t) = \mathcal{R}f(s,\varphi,t) = \int_{|\alpha|+|\beta|\le 1} d\beta\, d\alpha\, \delta(s-\beta\cos\varphi-\alpha\sin\varphi)\, f(\beta,\alpha,t)$$

Mathematical literature (a lot of) on the problem of computerized tomography:

$$\mathcal{R}f(s,\varphi,t) = \sum_{m=0}^{\infty}\sum_{l=0}^{m} g_{ml}(t)\, e^{i\,(-m+2l)\,\varphi}\, C_m^{\alpha}(s)$$

DGLAP to ERBL extension from the overlap

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Thm 2.3 *Let f be a compactly-supported locally summable function defined on $\mathbb{R}^2$ and $\mathcal{R}f$ its Radon transform. Let $(s_0, \omega_0) \in \mathbb{R} \times S^1$ and U_0 an open neighborhood of ω_0 such that:*

$$\text{for all } s > s_0 \text{ and } \omega \in U_0 \quad \mathcal{R}f(s,\omega_0) = 0\,. \qquad (2.101)$$

Then $f(\aleph) = 0$ on the half-plane $\langle \aleph | \omega_0 \rangle > s_0$ of $\mathbb{R}^2$. (See detailed proof on *p. 59**.)*

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Unicity: the knowledge of the GPD in DGLAP region allows the (unique) reconstruction of the GPD over the full range, $\xi, x \in R$, by capitalizing the polynomiality condition, and up to an ambiguity on the line $\beta = 0$.

... in progress!!!

Conclusions:

We just made a few modest steps in a very long way!!!

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- Computation of GPDs, DDs, PDFs, LFWFs and form factors in the **nonperturbative framework** of Dyson-Schwinger and Bethe-Salpeter equations.
- **Explicit check** of several theoretical constraints, including polynomiality, support property and soft pion theorem.
- Simple algebraic model exhibits **most features of the numerical solutions** of the Dyson-Schwinger and Bethe-Salpeter equations.
- **Very good agreement** with existing pion form factor and PDF data.
- In progress: *a priori* implementation of polynomiality and positivity.