CHAPTER 26

Statistical Fluctuations in Nuclear Processes

It is well recognized that we can never measure any physical magnitude exactly, i.e., with no error. Progressively more elaborate experimental or theoretical efforts result only in reducing the possible error of the determination. In reporting the result of any measurements it is therefore obligatory to specify also the probability that the result is in error by some specified amount, since *a gamble on relative correctness is always involved in all physical determinations.* The theory of statistics and fluctuations, summarized here, describes the mathematical procedure involved in the reduction of data, particularly data of the type encountered in nearly every measurement in nuclear physics.

Nuclear processes, in common with all microscopic processes, are random in ultimate character. Because of the relatively large energies released in nuclear processes, it is possible to study single random events. The application of statistical theory to such measurements is therefore doubly important because it contributes to our understanding of nuclear processes and it gives insight into the statistical distributions which describe other random processes whose individual events are not observable. The exponential decay distribution is an example of a result derivable solely from probability considerations (Chap. 15), without detailed knowledge of the mechanism involved.

In any series of measurements, the frequency of occurrence of particular values is expected to follow some "probability distribution law," or "frequency distribution." There are about a half dozen distributions which are used most often in the statistical appraisal and interpretation of nuclear data. We begin by discussing the four most fundamental of these frequency distributions. Later the very useful generalized Poisson distribution (Sec. 3) and the generalized interval distribution (Chap. 28, Sec. 2) will be considered.

The theory presented here is called *efficient statistics* for it extracts the maximum amount of statistical information from the data. A recent development termed *inefficient statistics* can extract a major portion, but not all, of the information by very much simpler calculations. It does this by making reasonable approximations in the efficient theory. It is thus necessary to understand the efficient theory presented here and in Chaps. 27 and 28 in order to be able to use the inefficient theory wisely. For this reason, only the efficient theory is treated here; however, the reader will find some useful inefficient statistics in Appendix G.

1. *Frequency Distributions*

a. The Binomial Distribution. The binomial distribution is the fundamental frequency distribution governing random events. The other frequency distributions can be derived from it.† Historically, it was the first probability distribution to be enunciated theoretically. Bernoulli, early in the eighteenth century, showed that if p is the probability that an event will occur, and $q = 1 - p$ is the probability that it will not occur, then in a random group of z independent trials the probability P_x that the event will occur x times is represented by that term in the binomial expansion of $(p + q)^z$ in which p is raised to the x power. Thus the expansion of $(p + q)^z$, which is always equal to unity, represents the sum of the individual probabilities of observing $x = z$ events, $x = (z - 1)$ events, . . . , $x = 0$ events, as follows:

$$\begin{aligned}(p + q)^z &= p^z + zp^{z-1}q + \frac{z(z-1)}{2!} p^{z-2}q^2 + \cdots + q^z \\ &= p^z + zp^{z-1}(1 - p) + \frac{z(z-1)}{2!} p^{z-2}(1 - p)^2 + \cdots + (1-p)^z \\ &= P_z + P_{z-1} + P_{z-2} + \cdots + P_0 = 1\end{aligned}$$

Any individual term in this binomial expansion can be written as

$$P_x = \frac{z!}{x!(z-x)!} p^x(1 - p)^{z-x} \tag{1.1}$$

which is the general form of the binomial distribution. The binomial distribution, Eq. (1.1), contains the two independent parameters p and z and rigorously applies to those phenomena in which the total number of trials z and the number of successes x are both *integers*.

It therefore describes the fluctuations in counting α rays from radioactive bodies, provided that p, which is equivalent to the probability $\lambda\,\Delta t$ that a particular atom will decay during an observation of short duration Δt, is constant. Like the normal and Poisson distributions, to be considered next, it represents the true probability when the total amount of radioactive material is essentially unaltered during the

† Representative treatises containing detailed proofs of many of the statements in this chapter include:

T. C. Fry, "Probability and Its Engineering Uses," D. Van Nostrand Company, Inc., New York, 1928.

R. A. Fisher, "Statistical Methods for Research Workers," Oliver & Boyd, Ltd., Edinburgh and London, 1930.

S. S. Wilks, "Mathematical Statistics," Princeton University Press, Princeton, N.J., 1943.

P. G. Hoel, "Introduction to Mathematical Statistics," 2d ed., John Wiley & Sons, Inc., New York, 1954.

N. Arley and K. R. Buch, "Introduction to the Theory of Probability and Statistics," John Wiley & Sons, Inc., New York, 1950.

period of the observations. The tests must, therefore, be made in a time interval Δt which is very short compared with the half-period of the radioactive substance. But under this restriction of small p, Poisson's distribution is a satisfactory approximation to the binomial distribution.

Applications of the binomial distribution to the tossing of coins and the throwing of dice are doubtless familiar to the reader. Here, it applies rigorously because p is constant. Thus the chance of throwing three, two, one, or zero aces in three throws of a single die (or in one throw of three dice) is

$$\begin{aligned}(\tfrac{1}{6} + \tfrac{5}{6})^3 &= (\tfrac{1}{6})^3 + 3(\tfrac{1}{6})^2(\tfrac{5}{6}) + 3(\tfrac{1}{6})(\tfrac{5}{6})^2 + (\tfrac{5}{6})^3 \\ &= \tfrac{1}{216} + \tfrac{15}{216} + \tfrac{75}{216} + \tfrac{125}{216} \\ &= P_3 + P_2 + P_1 + P_0 = 1\end{aligned}$$

Note that the chance of getting no ace in three throws is $\frac{125}{216} = 58$ per cent, although the average number of aces is $pz = 0.50$.

The binomial distribution is a special case of the multinomial distribution describing processes in which several results having fixed probabilities $p_1, p_2, \ldots, p_s$ are possible. The separate probabilities are then given by terms of the expansion $(p_1 + p_2 + \cdots + p_s)^z$, where $p_1 + p_2 + \cdots + p_s = 1$.

b. The Normal Distribution. The normal distribution† is an analytical approximation to the binomial distribution when z is very large. It is applicable to distributions in which the observed variable is not confined to integer values but can take on any value from $-\infty$ to $+\infty$. The normal distribution thus generally applies to a continuously variable observed magnitude, such as the distance separating two spectral lines, while the binomial and Poisson distributions are applied to discontinuous variables, such as particle counting rates, which take on successive whole-number integral values. The statistical theory of errors (D19) is ordinarily based on the normal distribution.

Near the center of the distribution curve the binomial distribution, for large z and constant average value $m = pz$, approaches identity with the normal distribution, which states that the probability dP_x that x will lie between x and $x + dx$ is

$$dP_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}\, dx \qquad (1.2)$$

where $e = 2.7183$ is the base of the natural system of logarithms, m is the true value of the quantity whose measured values are x, and σ is the *standard deviation*, a parameter which describes the breadth of the distribution of *deviations* $(x - m)$ from the mean. The standard deviation is discussed in detail in Sec. 2, but for the present it may be regarded simply as one of the two parameters, m and σ, of the normal distribution.

Figure 1.1 illustrates the general form of the normal distribution, drawn for a mean value of $m = 100$ and a standard deviation of $\sigma = 10$.

† The normal-distribution curve is sometimes erroneously referred to as the Gaussian error curve, but its derivation by Gauss (1809) was antedated by those of Laplace (1774) and DeMoivre (1735).

The ordinates are normalized so that the total area under the curve is unity. Thus the area included between any two abscissas x_1 and x_2 is the probability that a single measurement of x will lie between x_1 and x_2, while a very large number of measurements of x would have a mean value of m. The correspondence between Fig. 1.1 and Eq. (1.2) lies in the relationships

$$dP_x = dA = y\,dx \qquad \int_{-\infty}^{\infty} dP_x = A = 1 \tag{1.3}$$

in which y is the ordinate of Fig. 1.1 and dA is an element of area. The coefficient $1/\sigma\sqrt{2\pi}$ in Eq. (1.2) normalizes the area to unity, as given by Eq. (1.3).

Differentiation of Eq. (1.2) shows that the points of maximum slope, at which $d^2y/dx^2 = 0$, fall at the points $(x - m) = \pm\sigma$, where the slope

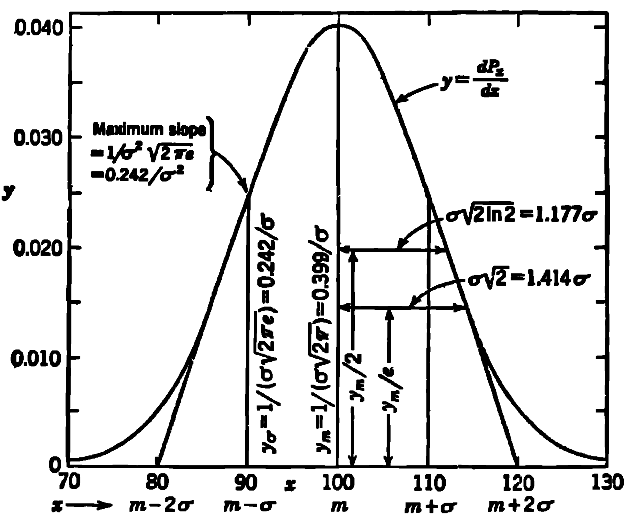


Fig. 1.1 Normal distribution for the special case of a mean value $m = 100$ and a standard deviation $\sigma = 10$.

has the value $1/\sigma^2\sqrt{2\pi e}$. Tangents to the distribution curve at these inflection points intersect the x axis at $(x - m) = \pm 2\sigma$. The ratio of the ordinate y_σ at these symmetrical points of maximum slope to the maximum ordinate $y_m = 1/\sigma\sqrt{2\pi}$ at $x = m$ is $y_\sigma/y_m = e^{-\frac{1}{2}} = 0.6065$. The half width is $\sigma\sqrt{2\ln 2} = 1.177\sigma$ at $y = y_m/2$ (half maximum) and is $\sigma\sqrt{2} = 1.414\sigma$ at $y = y_m/e$ ($1/e$ of maximum). These geometrical relationships offer a convenient method of determining σ graphically from an experimentally determined distribution curve.

Figure 1.2 gives the results of integration of the normal distribution between various limits; from it can be read the chance that a single observation of x will differ from the mean value m by more than any arbitrary assigned amount. Figure 1.2 has many other uses which will be referred to later.

c. The Poisson Distribution. Poisson's distribution *describes all random processes whose probability of occurrence is small and constant.* It therefore has wide and diverse applicability and describes the statistical fluctuations in such random processes as the number of soldiers kicked and killed yearly by cavalry horses, the disintegration of atomic nuclei, the emission of light quanta by excited atoms, and the appearance of

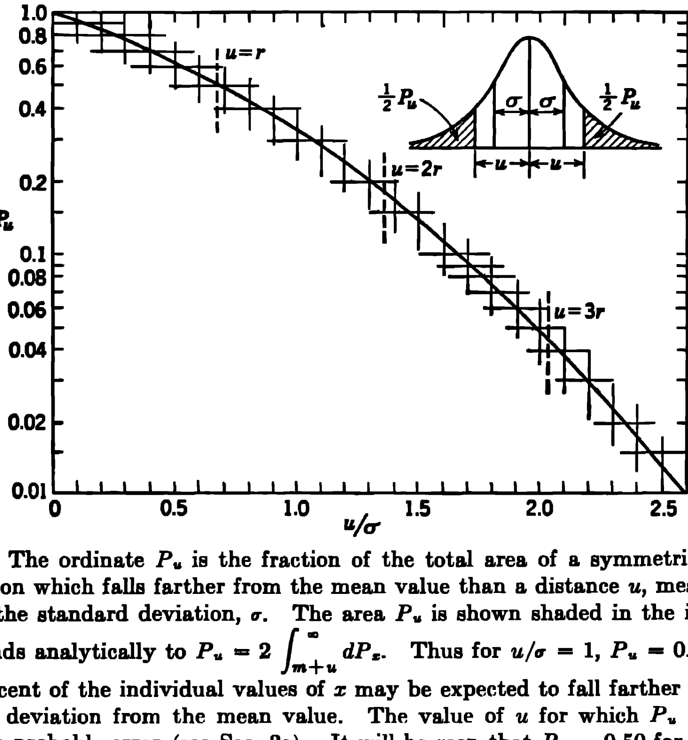


Fig. 1.2 The ordinate P_u is the fraction of the total area of a symmetric normal distribution which falls farther from the mean value than a distance u, measured in units of the standard deviation, σ. The area P_u is shown shaded in the inset and corresponds analytically to $P_u = 2\int_{m+u}^{\infty} dP_x$. Thus for $u/\sigma = 1$, $P_u = 0.317$, and 31.7 per cent of the individual values of x may be expected to fall farther than one standard deviation from the mean value. The value of u for which $P_u = 0.50$ is called the probable error (see Sec. 2e). It will be seen that $P_u = 0.50$ for $u = r = 0.6745\sigma$. Particular numerical values which find frequent use are

u/σ	0.5	0.6745	1	1.349	2	2.024	2.698	3
P_u	0.617	0.500	0.317	0.178	0.0455	0.0431	0.00706	0.00272

cosmic-ray bursts. The Poisson distribution applies to substantially all observations made in experimental nuclear physics.

The Poisson distribution can be deduced as a limiting case of the binomial distribution, for those random processes in which the probability of occurrence is very small, $p \ll 1$, while the number of trials z becomes very large and the mean value $m = pz$ remains fixed. Then in Eq. (1.1) $m \ll z$ and $x \ll z$, and so we can write, approximately,

$$\frac{z!}{(z-x)!} \simeq z^x$$

$$(1-p)^{z-x} \simeq e^{-p(z-x)} \simeq e^{-pz}$$

and in the limiting case of small probability p, Eq. (1.1) approaches

$$P_x = \frac{z^x p^x}{x!} e^{-pz} = \frac{m^x}{x!} e^{-m}$$

which is the Poisson distribution.

A much clearer feeling for the statistical principles underlying the Poisson distribution is obtained by deriving this frequency distribution from first principles. Specializing the general conditions under which the Poisson distribution holds to the readily visualized case of a radioactive disintegration, we would write the following necessary and sufficient conditions:

1. The chance for an atom to disintegrate in any particular time interval is the same for all atoms in the group (all atoms identical).
2. The fact that an atom has disintegrated in a given time interval does not affect the chance that other atoms may disintegrate in the same time interval (all atoms independent).
3. The chance for an atom to disintegrate during a given time interval is the same for all time intervals of equal size (mean life long compared with the total period of observation).
4. The total number of atoms and the total number of equal time intervals are large (hence statistical averages significant).

Let a be the average rate of appearance of particles from such a random process; then the average number of events in a time interval t is at. Then in a short time interval, dt, such that $a\,dt \ll 1$, the quantity $a\,dt$ is simply the probability $P_1\,(dt)$ of observing one particle in the time dt. As dt decreases without limit, the probability of observing two or more particles in the time dt becomes vanishingly small in comparison with the probability of observing one particle, that is, $P_1\,(dt) \gg P_2\,(dt) \gg P_3\,(dt)$ . . . . The probability of observing no particle in dt is

$$P_0\,(dt) = 1 - P_1\,(dt) = 1 - a\,dt$$

We may now write the probability of observing x particles in the time $(t + dt)$ as the combined probabilities of $(x - 1)$ particles in t and one in dt, or of x particles in t and none in dt; thus

$$\begin{aligned} P_x(t + dt) &= P_1\,(dt)\,P_{x-1}(t) + P_0\,(dt)\,P_x(t) \\ &= a\,dt\,P_{x-1}(t) + (1 - a\,dt)\,P_x(t) \end{aligned} \tag{1.4}$$

Rewriting Eq. (1.4) in differential form, we have

$$\begin{aligned} \frac{dP_x(t)}{dt} &= \frac{P_x(t + dt) - P_x(t)}{dt} \\ &= a[P_{x-1}(t) - P_x(t)] \end{aligned} \tag{1.5}$$

The solution (B19) of Eq. (1.5) is

$$P_x(t) = \frac{(at)^x}{x!} e^{-at} \tag{1.6}$$

as can be verified by differentiation. Now if the equal intervals of time are chosen of length t, then the average number of particles per interval is $at = m$; substituting this in Eq. (1.6), we have the usual form of the Poisson distribution

$$P_x = \frac{m^x}{x!} e^{-m} \tag{1.7}$$

in which P_x is the probability of observing x events when the average for a large number of tries is m events. Although m may have any positive value, x is restricted to integer values only. It is easy to show that Eq. (1.7) correctly leads to

$$\sum_{x=0}^{x=\infty} P_x = 1$$

It must be noted that, in contrast with the two previous frequency distributions, *the Poisson distribution has but one parameter m.* The binomial distribution with parameters p and z becomes identical with the Poisson distribution when $p \to 0$ in such a way that $zp = m$. The normal distribution, near the center of the distribution, approaches equality with Poisson's distribution when m is large, so that the histogram of Poisson's discontinuous distribution approaches the continuous normal distribution (compare Figs. 1.1 and 1.3).

Following the numerical evaluation† of a particular value of P_x, other neighboring values may be computed quickly by using the following exact relationships, which can be derived easily from Eq. (1.7)

$$P_{x-1} = \frac{x}{m} P_x \tag{1.8}$$

$$P_{x+1} = \frac{m}{x+1} P_x \tag{1.9}$$

$$P_0 = e^{-m} \tag{1.10}$$

The Poisson distribution is slightly asymmetric, favoring low values of x. Thus substitution of $x = m$ in Eq. (1.8) shows that if the mean value is an integer *the probability of observing one less than the mean value is the same as the probability for the mean value.*

In computations with Eq. (1.7) it is often convenient to use Stirling's approximation to the factorial

$$x! = \sqrt{2\pi x}\, x^x e^{-x} \left(1 + \frac{1}{12x} + \cdots\right) \tag{1.11}$$

in which neglect of the final parentheses involves a negative error of only

† Extremely useful tables of the individual terms P_x and especially of the cumulated terms $\sum_x^\infty P_x$ for Poisson distributions with $m = 0.001$ to 100 have been published by E. C. Molina (M49).

0.8 per cent for $x = 10$, and 0.08 per cent when $x = 100$. Then Eq. (1.7) becomes, to a good approximation, for $x > 10$

$$P_x \simeq \frac{1}{\sqrt{2\pi x}} \left(\frac{m}{x}\right)^x e^{-(m-x)} \tag{1.12}$$

The probability of actually observing the mean value m in a series of observations on a random process of constant average value is surprisingly small. This is seen by substitution of $x = m$ in Eq. (1.12) which gives the following values

$m =$	10	100	1,000
$P_m =$	0.127	0.040	0.013

The Poisson distribution must always be represented by a histogram, since x must assume whole-number values only. Figure 1.3 illustrates the Poisson distribution for $m = 100$; the slight asymmetries should be noted, as well as the similarity with the symmetric normal distribution

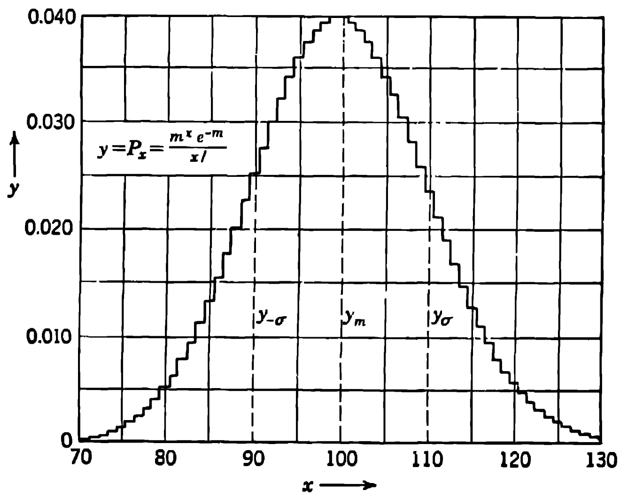


Fig. 1.3 Poisson distribution for $m = 100$. Note that $P_{90} > P_{110}$, whereas $P_{70} < P_{130}$; also that $P_{99} = P_{100}$, illustrating the asymmetries of the Poisson distribution. The envelope of this histogram is very similar to the normal distribution shown in Fig. 1.1 only because of our arbitrary choice of $\sigma = \sqrt{m}$ in Fig. 1.1. The standard deviation, σ for the Poisson distribution, Eq. (2.7), is always $\sqrt{m}$, but σ is an independent parameter in the normal distribution.

of Fig. 1.1. For small values of m, say, between $m = 1$ and $m = 10$, the Poisson distribution is very asymmetric and is not well approximated by the normal distribution.

The Poisson frequency distribution treats all the *intervals* as *independent;* this restriction in application is removed by the interval distribution.

d. The Interval Distribution. The interval distribution is derived from Poisson's distribution and describes the distribution in size of the time intervals between successive events in any random process in which the mean rate has the constant value of a events per unit time (M12, R20). From Eq. (1.7) or (1.10) the probability that there will be no events in a time interval t, during which time there should be at events on the average, is†

$$P_0 = \frac{(at)^0}{0!} e^{-at} = e^{-at} \tag{1.13}$$

The probability that there will be an event in the time interval dt is simply $a\,dt$. The combined probability that there will be no events during the time interval t, but one event between time t and $t + dt$, is $ae^{-at}\,dt$. Hence, in a random distribution which follows the Poisson distribution and has a constant average interval of $1/a$, the probability dP_t that the duration of a particular interval will be between t and $t + dt$ is

$$dP_t = ae^{-at}\,dt \tag{1.14}$$

We see at once that *small time intervals have a higher probability* than large time intervals between the randomly distributed events.

If the data concern a large number N of intervals, then the number of intervals greater than t_1 but less than t_2 is

$$\begin{aligned} n &= N \int_{t_1}^{t_2} ae^{-at}\,dt \\ &= N(e^{-at_1} - e^{-at_2}) \end{aligned} \tag{1.15}$$

where a is the average number of events per unit time. Equations (1.14) and (1.15) are the general differential and integral forms of the interval distribution for randomly spaced events.

Two limiting cases are of special interest. Letting $t_2 \to \infty$, we find that *the number of intervals greater than any duration t is Ne^{-at}*, in which at is simply the average number of events in the interval t. Because the average interval is $\bar{t} = 1/a$, we note that the fraction of the intervals which are longer than the average is $n/N = e^{-1} = 0.37$.

Letting $t_1 \to 0$, Eq. (1.15) shows that *the number of intervals shorter than any duration t is $N(1 - e^{-at})$*. Examples of the usefulness of the interval distribution in α-ray counting experiments and in cosmic-ray-burst observations will be given in Chap. 27, Sec. 4. A generalization of the interval distribution, giving the frequency distribution of intervals which contain any predetermined number of random events, is derived in Chap. 28, Sec. 2.

Problems

"The reader is, however, advised that the detailed working of numerical examples is essential to a thorough grasp, not only of the technique, but of the principles by which an experimental procedure may be judged to be satisfactory"

† That factorial zero equals unity follows from the gamma functions:

$$n! = \Gamma(n + 1) \qquad \Gamma(1) = 1 \qquad \Gamma(0 + 1) = 1$$

and effective." (R. A. Fisher, in the preface to the first edition of his "The Design of Experiments," Oliver & Boyd, Ltd., Edinburgh and London, 1935.)

1. From elementary probability arguments and consideration of the number of combinations and permutations of z things taken x at a time, "derive" the binomial distribution.

2. In 1693 (hence pre-Bernoulli and pre-binomial distribution), Samuel Pepys propounded the following question to his friend Isaac Newton, who prepared a lengthy response and engaged the tax accountant George Tollet in a protracted controversy over the answer:

"*A* has 6 dice in a box, with which he is to fling a six, *B* has in another box 12 dice, with which he is to fling 2 sixes, *C* has in another box 18 dice, with which he is to fling 3 sixes. Question—whether *B* and *C* have not as easy a task as *A* at even luck."

(*a*) Assuming Pepys meant *exactly* one, two, and three sixes for the three contestants, what are their chances of succeeding on a single throw?

(*b*) Assuming he meant *at least* one, two, and three sixes, in what direction will this modify their chances of success?

3. With a simultaneous throw of six dice, calculate the probabilities of obtaining just zero, one, two, three, four, five, and six sixes. Show that the sum of these probabilities is unity when the solutions are obtained using the binomial distribution. Compare with these the probabilities for the same events as given by the Poisson distribution. Point out the reasons for these differences.

4. On June 5, 1951, Dom DiMaggio had a batting average of 0.359, had been at bat 189 times in 44 games, and had hit safely at least once in each of his last 25 consecutive games.

(*a*) What is the probability that he will hit safely if he is at bat four times in the baseball game on June 6, 1951?

(*b*) What is the probability of his hitting safely in 26 consecutive games, if he is at bat four times in each game?

(*c*) Explain concisely why the odds in (*a*) and (*b*) are so different.

5. Consider the chances of a bomber pilot surviving a series of statistically identical raids, in which the chance of being shot down is always 5 per cent.

(*a*) From an original group of 1,000 such pilots, how many should survive 1, 5, 10, 15, 20, 40, 80, and 100 raids? Plot the survival curve, with the number of flights as abscissa.

(*b*) Estimate the mean life of a pilot in number of raids.

(*c*) In a single raid of 100 planes, what are the chances that zero, one, five, or ten planes will be lost?

6. Calculate and plot a normal distribution having a mean value of 10 and a standard deviation of 3.

7. Show that the sum of the probabilities $P_x = e^{-m}m^x/x!$ of all possible positive values $x = 0, 1, 2, \ldots$ is unity for the Poisson distribution.

8. In any Poisson distribution, show analytically that the probability of observing one less than the mean value is the same as the probability for the mean value.

9. In any Poisson distribution,

(*a*) Show that the ratio of the probability P_{2m} of observing twice the mean value to the probability P_m of observing the mean value is

$$P_{2m}/P_m = m^m m!/(2m)!$$

or when $m \gg 1$,

$$P_{2m}/P_m = 0.707(0.824)^{2m}$$

(*b*) Compare P_{2m}/P_m for $m = 2, 10, 100$.

10. Calculate and plot a Poisson distribution with a mean value of 10 and values of x from 0 to 20. Compare with the normal distribution for $m = 10$, $\sigma = 3$.

11. The average background of a certain α-ray counter is 20 α rays per hour. In how many hours out of 200 would you expect to observe only 10 α rays?

12. In an α-ray counting experiment, on a source of constant average intensity, a total of 19,278 α rays are counted in 51 hr of continuous observations. The time of arrival of each α ray is recorded on a tape, so that the number of α rays recorded in each successive 1-min interval can be determined.

(*a*) What is the average number of α rays per 1-min interval?

(*b*) In how many of the total number of 1-min intervals would you expect to observe no α rays?

(*c*) In how many 1-min intervals should one observe one α ray?

(*d*) In how many 1-min intervals should one observe six α rays?

13. A Geiger-Müller counter having a resolving time of 300 μsec is placed in a plane parallel beam of 5-Mev photons from a pulsed generator. The counter has a cylindrical cathode 2 cm in diameter and 10 cm long, is placed with its axis perpendicular to the beam, and has an absolute efficiency of 2 per cent for 5-Mev photons. Each pulse of photons from the generator is 50 μsec in duration, and the repetition rate is 120 pulses per second. Under these conditions of operation, the counter displays an average counting rate of 3,600 counts per minute. During each pulse, what is the flux in photons per second per square centimeter at the counter position?

14. In a random distribution having an average interval $\bar{t}$, show by application of the interval distribution that the average value of the *absolute* deviations from the mean interval $\bar{t}$ is

$$|t - \bar{t}|_{av} = \frac{2}{e}\,\bar{t} = 0.7358\bar{t}$$

15. The average background of a certain α-ray counter is 30 α rays per hour.

(*a*) What fraction of the intervals between successive counts will be longer than 5 min?

(*b*) What fraction of the intervals will be longer than 10 min?

(*c*) What fraction of the intervals will be shorter than 30 sec?

16. A certain radioactive sample contains a mixture of an α-ray emitter and a β-ray emitter. The two substances are assumed to be independent. Using a particular pair of counters, the observed activities are A α counts per minute and B β counts per minute.

(*a*) What is the combined probability that a particular interval between two successive α rays will have a duration between t and $t + dt$ and will also contain exactly x β rays?

(*b*) Show that the probability $M(x)$ of observing just $x(x = 0, 1, 2, \ldots)$ β rays in the time interval between *any* two successive α-ray counts is

$$M(x) = \frac{R^x}{(R + 1)^{x+1}}$$

where $R = B/A$.

(*c*) What is the probability of observing just one α ray in the time interval between *successive β rays* if $A = 100$ and if $B = 500$ counts per minute?

17. Derive an interval distribution governing the output of a scale-of-2 circuit if the input receives randomly distributed pulses at an average rate of a pulses per minute. Specifically, (*a*) Show that the number n of observed intervals of length between t_1 and t_2 min, when N is the total number of scale-of-2 intervals studied, is

$$\frac{n}{N} = (at_1 + 1)e^{-at_1} - (at_2 + 1)e^{-at_2}$$

(*b*) Derive an expression for the fraction n/N of the intervals which will be longer than t.

(*c*) Derive an expression for the fraction of the intervals which will be shorter than t.

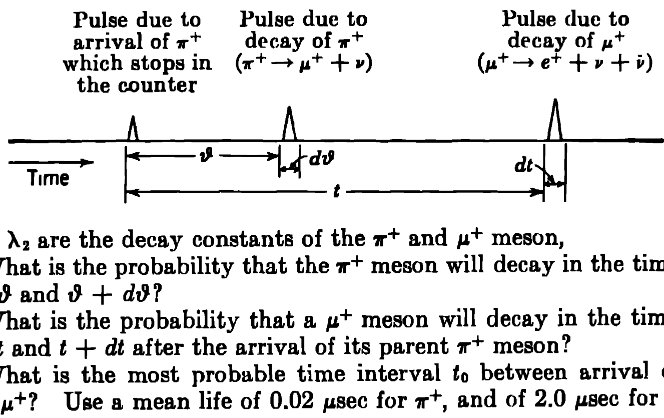
18. A scale-of-2 counter gave 292 pulses in 11 hr.

(*a*) What are the duration of the average interval and the average rate of the statistical process (scale of 1)?

(*b*) Compute the number of scale-of-2 intervals expected to be longer than 7.5 min. (Four were observed.)

(*c*) Compute the number of scale-of-2 intervals expected to be shorter than 5 sec. (One was observed.)

19. An interval distribution is to be derived, which will describe the distribution of time intervals between the arrival of π^+ mesons in a counter and the decay of the daughter μ^+ meson. The events to be considered are:



If λ_1 and λ_2 are the decay constants of the π^+ and μ^+ meson,

(*a*) What is the probability that the π^+ meson will decay in the time interval between ϑ and $\vartheta + d\vartheta$?

(*b*) What is the probability that a μ^+ meson will decay in the time interval between t and $t + dt$ after the arrival of its parent π^+ meson?

(*c*) What is the most probable time interval t_0 between arrival of π^+ and decay of μ^+? Use a mean life of 0.02 μsec for π^+, and of 2.0 μsec for μ^+.

2. *Statistical Characterization of Data*

a. Mean Value. In any *finite* series of measurements we can never find the *exact* value of the true mean value m, which corresponds to the infinite population of (i.e., an infinite amount of) data. Although the mean value† is constant, our individual measurements should be distributed about this mean value in a manner given by the particular frequency distribution which describes the process being studied. For the

† The *mean* value (i.e., the average value) is to be distinguished from the *modal* value (i.e., the most probable value) and from the *median* value (i.e., the value which is as frequently exceeded as not). Only for a symmetric distribution are the mean, mode, and median coincident. For an asymmetric distribution of numbers such as: 2, 3, 5, 6, 7, 8, 8, 9, 9, 9, 11, 11, 12, the mean value is 7.69, the mode is 9, and the median is 8.

four frequency distributions discussed in the previous section, it can be shown that our best approximation to m is simply the arithmetic average $\bar{x}$ of the n separate measurements, $x_1, x_2, x_3, \ldots, x_n$; that is,

$$m \simeq \bar{x} = \frac{1}{n}\sum_{1}^{n} x_i \tag{2.1}$$

The "expectation value" of any statistical variable is synonymous with the mean value obtained for this variable in a large number of trials. Thus the *expectation value* of x is m.

b. Standard Deviation and Variance. The breadth of the statistical fluctuations of our individual readings about the true mean value is expressed quantitatively by the fundamentally important parameter, the *standard deviation* σ. For a particular mean value m, a small σ gives a sharply peaked distribution, whereas a large σ gives a broad, flattened distribution. In any case, the significance of the standard deviation as descriptive of the spread of the data is best seen in the normal distribution. Figure 1.2 shows that, in the normal distribution, about 32 per cent of a large series of individual observations must deviate from the mean value by more than $\pm\sigma$ and consequently that 68 per cent of the individual observations should lie within the band $(\bar{x} \pm \sigma)$.

For any frequency distribution, the standard deviation (often abbreviated S.D.) is defined as the square root of the average value of the square of the individual deviations $(x - m)$, for a large number of observations. Thus

$$\sigma^2 = \sum_{x=-\infty}^{x=\infty} (x - m)^2 P_x \tag{2.2}$$

or, in terms of a large series of n measurements of x,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{i=n} (x_i - m)^2 \tag{2.3}$$

The square of the S.D. is thus seen to be simply the second moment of the frequency distribution taken about the mean. The quantity σ^2 is usually called the *variance*. As is suggested by the form of Eq. (2.3), σ occasionally is called the *root-mean-square error*.

We can now use Eq. (2.2) for the derivation of analytical expressions for the S.D. of the various distributions.

For the binomial distribution, with mean value $m = zp$, the square of the standard deviation is

$$\sigma^2 = \sum_{x=0}^{x=z} (x - zp)^2 P_x = \sum_{x=0}^{x=z} \frac{(x - zp)^2 z! p^x (1 - p)^{z-x}}{x!(z - x)!}$$

Upon expansion and summation, this expression reduces to simply

$$\sigma^2 = zp(1 - p) \tag{2.4}$$

or, because the mean value of x is $m = zp$, we have also

$$\sigma^2 = m(1 - p) \tag{2.5}$$

Note especially that, for the binomial distribution, the variance σ^2 is always less than the mean value m ($\simeq \bar{x}$).

For the normal distribution, the evaluation of Eq. (2.2) by integration gives, of course,

$$\sigma^2 = \int_{-\infty}^{+\infty} (x - m)^2 \, dP_x = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{+\infty} (x - m)^2 e^{-(x-m)^2/2\sigma^2} \, dx = \sigma^2 \tag{2.6}$$

because the S.D. is simply one of the two independent parameters of the normal distribution and therefore may have any value.

For the Poisson distribution, however, the S.D. has a definite value in terms of the mean value, which is the only parameter of the Poisson distribution. From Eq. (2.2), we find on expansion that

$$\sigma^2 = \sum_{x=0}^{x=\infty} \frac{(x - m)^2 m^x}{x!} e^{-m} = m \tag{2.7}$$

Hence for the Poisson distribution, the S.D. of the distribution of individual observations is simply $\sqrt{m}$. This result is, of course, in agreement with the S.D. of the binomial distribution, Eq. (2.5), in the limiting case for $p \ll 1$.

For the interval distribution governing randomly distributed events occurring at an average rate a, hence with average interval $\bar{t} = 1/a$, Eqs. (1.14) and (2.2) lead to

$$\sigma^2 = \int_0^\infty \left(t - \frac{1}{a}\right)^2 ae^{-at} \, dt = \left(\frac{1}{a}\right)^2 \tag{2.8}$$

Thus the S.D. is just equal to the average interval $1/a$.

Table 2.1 now summarizes the properties of the four frequency distributions which apply to random processes having a constant average value.

c. Estimate of Standard Deviation from a Finite Series of Observations. In a finite series of n observations, we can never know m exactly. Hence we can never determine σ exactly, as implied in Eq. (2.2) which applies to the infinite population of data. Our best approximation to the S.D. of the distribution, in terms of our finite number n of observations, can be shown to be

$$\sigma^2 \simeq \frac{n}{n - 1} \left[\frac{1}{n} \sum_1^n (x_i - \bar{x})^2\right] = \frac{1}{n - 1} \sum_1^n (x_i - \bar{x})^2 \tag{2.9}$$

TABLE 2.1. SUMMARY OF FREQUENCY DISTRIBUTIONS DESCRIBING RANDOM PROCESSES

For ease of reference we include two generalizations which are to be described in Sec. 3 of this chapter and in Chap. 28, Sec. 2.

Frequency distribution	Distribution	Average value	Standard deviation	Distribution is approximation to binomial when
Binomial	$P_x = \frac{z!}{x!(z-x)!} p^x(1-p)^{z-x}$	$\bar{x} = zp$	$\sqrt{zp(1-p)}$	
Normal	$dP_x = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}\,dx$	$\bar{x} = m$	σ	$z \gg 1$
Poisson	$P_x = \frac{m^x}{x!} e^{-m}$	$\bar{x} = m$	$\sqrt{m}$	$p \ll 1;\ pz = m$
Interval	$dP_t = ae^{-at}\,dt$	$\bar{t} = \frac{1}{a}$	$\frac{1}{a}$	
Generalized Poisson, this chapter, Sec. 3	$P_u = aP_x + bP_y + cP_z + \cdots$	$u = ax + by + cz + \cdots$	$\sigma = \sqrt{a^2x + b^2y + c^2z + \cdots}$	
Generalized interval, Chap. 28, Sec. 2	$dP_t = \frac{a^s t^{s-1}\,e^{-at}}{(s-1)!}\,dt$	$\bar{t} = \frac{s}{a}$	$\frac{\sqrt{s}}{a}$	

This practical expression for the (S.D.)^2 differs from Eq. (2.3) only in its denominator $(n - 1)$ and in the use of $\bar{x}$ in place of m. The term $(n - 1)$ is to be correlated with the view that the dispersion among the data is associated with the number of "degrees of freedom." From n independent observations of x we are provided originally with n independent equations. We reduce this number by one when we compute $\bar{x}$ and hence have only $(n - 1)$ independent data from which to compute σ. It can be seen readily that, in the special case in which only one observation is made, $\bar{x} = x$, and σ is indeterminate. The latter condition is correctly given by Eq. (2.9) but could not be obtained from Eq. (2.3) directly.

In the theory of mathematical statistics the so-called "sample variance" s^2 is defined as

$$s^2 \equiv \frac{1}{n} \sum_{1}^{n} (x_i - \bar{x})^2 \tag{2.10}$$

It can be shown quite generally that the *expectation value* $E[s^2]$ for the *sample variance* of n observations is

$$E[s^2] = \frac{n - 1}{n} \sigma^2 \tag{2.11}$$

This is the formal basis for our Eq. (2.9) which we shall use hereafter without further explicit reference to the sample variance.

d. Standard Deviation of the Mean Value (Standard Error). If our n individual measurements of x exhibit, say, an approximately normal distribution about the mean value $\bar{x}$, then Fig. 1.2 shows that some 68 per cent of our individual observations have fallen within the central band $\bar{x} \pm \sigma$. This means that *one* additional single observation, if made, would have a 68 per cent chance of lying within $\bar{x} \pm \sigma$. In recognition of this probability interpretation, the "*standard deviation of the distribution*" σ, as determined from Eq. (2.3) or (2.9), can be called more precisely the "*standard deviation of a single observation.*"

Obviously, if we, or another observer, were to repeat our entire experiment of n observations, we should expect the new mean value to have much greater than a 68 per cent chance of falling within $\bar{x} \pm \sigma$. Therefore, in reporting our mean value $\bar{x}$, we wish to assign to it a S.D. of the mean value $\sigma_{\bar{x}}$ such that there is approximately a 68 per cent chance that some new mean value $\bar{x}_2$ will lie within the band $(\bar{x} \pm \sigma_{\bar{x}})$. Obviously, $\sigma_{\bar{x}}$ is smaller than σ.

It is well known in the theory of errors that a series of k mean values, $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_k$, each based on n observations, will tend to exhibit a normal distribution about their grand average $\bar{\bar{x}}$. This is true if n is sufficiently large, even if the parent population x_i is not normally distributed but is, for example, an asymmetric Poisson distribution. In general, *the distribution of mean values tends to be much more nearly normal than the parent population.* This is the justification for a theoretical derivation of the relationship between σ and $\sigma_{\bar{x}}$, based on a normal distribution of $\bar{x}$. Then it can be shown that in a large series of k measurements of the mean value $\bar{x}$, each based on n measurements of x, the grand average $\bar{\bar{x}}$ approaches the true mean m and that the S.D. of $\bar{x}$ depends upon n in the following way

$$\sigma_{\bar{x}}^2 = \frac{1}{k}\sum_{j=1}^{j=k} (\bar{x}_j - m)^2 = \frac{\sigma^2}{n} \tag{2.12}$$

The result of a single series of n measurements of x is then to be reported as $(\bar{x} \pm \sigma_{\bar{x}})$ where

$$\bar{x} = \frac{1}{n}\sum_{1}^{n} x_i$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

$$\simeq \sqrt{\frac{1}{n(n-1)}\sum_{1}^{n}(x_i - \bar{x})^2} \tag{2.13}$$

Then a repetition of the series of n measurements would, in general, give a different mean value, but the chance that the new mean value would lie within $(\bar{x} \pm \sigma_{\bar{x}})$ is 68 per cent. The S.D. of the mean value $\sigma_{\bar{x}}$ is often called the *standard error.*

The validity of Eq. (2.12) or (2.13) is almost self-evident for the Poisson distribution. Suppose a total of $v = \sum_{1}^{n} x_i$ random events are observed. Then by Eq. (2.7) the S.D. in this single observation is $\sqrt{\sum_{1}^{n} x_i}$, so that the result would be reported as

$$v \pm \sigma = \sum_{1}^{n} x_i \pm \sqrt{\sum_{1}^{n} x_i} \tag{2.14}$$

and the *fractional S.D.* would be

$$\text{F.S.D.} \equiv \frac{\sigma}{v} = \frac{1}{\sum_{1}^{n} x_i}\sqrt{\sum_{1}^{n} x_i} = \frac{1}{\sqrt{\sum_{1}^{n} x_i}} \tag{2.15}$$

Suppose now that a zealous assistant was present, while you tallied only the total number $\sum_1^n x_i$ events, and that he broke the data into n contiguous and equal intervals, recording

$$x_1 + x_2 + x_3 + \cdots + x_n = \sum_1^n x_i$$

Then he would obtain

$$\bar{x} = \frac{1}{n}\sum_1^n x_i$$

$$\sigma = \sqrt{m} \simeq \sqrt{\bar{x}}$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1}{n}\sqrt{\sum_1^n x_i}$$

and the result of the measurements would be reported as

$$\bar{x} \pm \sigma_{\bar{x}} = \frac{1}{n}\left(\sum_1^n x_i \pm \sqrt{\sum_1^n x_i}\right) \tag{2.16}$$

which has the same *fractional* S.D. as Eq. (2.15). In fact, Eq. (2.16) could have been obtained directly from Eq. (2.14) by simply dividing by the number of classifications n into which the data were subdivided. In either case, and in general, the observation of a total of $\sum_1^n x_i$ randomly distributed events has a *fractional* S.D. of $1 \Big/ \sqrt{\sum_1^n x_i}$. Thus the S.D. in counting 100 random events is 10 per cent, and one must count 10,000 events to reduce the S.D. to 1 per cent. *No mere method of treating the same total data can ever reduce the magnitude of the fractional uncertainty due purely to randomicity.*

e. Probable Error. A result quoted as $\bar{x} \pm \sigma_{\bar{x}}$ implies that the chance that the average value $\bar{x}$ differs from the true mean value m by more than $\sigma_{\bar{x}}$ is 0.317, if the error distribution is normal. While the S.D. has a definite statistical value and a basic significance in the principal frequency distributions, there has been an occasional tendency to fail to use it in reporting physical results. Instead, a quantity derived from the S.D. and called the probable error is often given. Its wide adoption rests on its easily visualized interpretation and perhaps also on the fact that, of all the common types of error specification, the probable error has the least value and hence makes the data look best.

The probable error is, by definition, *exactly as likely to be exceeded as not.* The probable error is ordinarily derived from the S.D. From Fig. 1.2 it can be seen that the particular error r which has exactly a 0.50

chance of being exceeded in a *normal distribution* is

$$r = 0.6745\sigma \qquad \text{and } r_{\bar{x}} = 0.6745\sigma_{\bar{x}} \tag{2.17}$$

Similarly, for a normal distribution, the chance that the actual error $(m - \bar{x})$ exceeds $r_{\bar{x}}$, $2r_{\bar{x}}$, $3r_{\bar{x}}$, etc. (without regard to sign) is given in the following table:

Chance that $\lvert m - \bar{x}\rvert$ is greater than...........	$r_{\bar{x}}$	$2r_{\bar{x}}$	$3r_{\bar{x}}$	$4r_{\bar{x}}$
Is........................	0.500	0.178	0.043	0.0071

Intermediate values may be read from Fig. 1.2. It is customary therefore to regard $3r_{\bar{x}}$ (or $2\sigma_x$) as equivalent to the *limit of error*, though this is clearly arbitrary and unreal rigorously. Moreover, the specification of a physical result as $\bar{x} \pm r_{\bar{x}}$ is exact *only* for a symmetric normal distribution.

Any asymmetry in the actual distribution will result in the probable positive error differing from the probable negative error for single observations; that is, ordinates of the distribution curve at $x = 0$, $(m - r)$, m, $(m + r)$, ∞, no longer divide the errors (area) into four equal parts (quartiles). Of course, an analogous objection can often be made to the lack of significance of the plus and minus sign if used with the S.D. of an asymmetric distribution. The only rigorous interpretation of S.D. of an asymmetric distribution is as root-mean-square error, Eqs. (2.9) and (2.13), and not as a plus or minus value having symmetric probabilities of being exceeded. It is only because the asymmetry of the Poisson distribution becomes small, and because this distribution approaches the normal distribution in the vicinity of the mean value when $m \gg 1$, that probable error can have any exact significance.

Graphical integration of Poisson distributions shows that the asymmetry is of the order of 10 to 4 per cent for $m = 10$ to 100 and vanishes as $m \to \infty$. The asymmetry for $m \geq 10$ does not invalidate Eq. (2.17), but for $m < 10$ much more significance attaches to the S.D. than to the probable error. The general dependence of r on $\sigma = \sqrt{x}$ for the Poisson distribution is given in the following table (R25):

x................	20	60	100	200	400	1,000	∞
$\frac{r}{\sigma} = \frac{r}{\sqrt{x}}$	0 575	0 613	0.628	0.640	0.647	0.660	0.6745

It will be noted that use of the conventional expression $r = 0.6745\sigma$ even for the Poisson distribution results in a conservative estimate of the probable error and is a safe procedure to follow.

If the mean value of an asymmetric distribution is estimated from a very large number n of observations, then the probable error of the mean value can have a true "plus-or-minus" significance, because the distribution of mean values is always more nearly normal than the parent population.

f. Dimensions of Statistical Parameters. From consideration of Eq. (2.7) or (1.7), it is evident that both x and σ must be *dimensionless* quantities, since σ has the same dimensions as x and $\sqrt{x}$. It is generally true that all such quantities in the distribution functions and in other statistical expressions are without dimensions. For example, $\bar{x}$ may be physically the average number of counts per minute, but statistically the time unit chosen is only an arbitrary interval or classification by which the data have been taken. It is to be regarded statistically as dimensionless. This can be visualized by considering time intervals measured off on a chronograph tape, in which case the particular interval used for classification might equally well be one second, or an equivalent length of tape, or even an equivalent mass of tape. The interval itself does not have the dimensions of time, length, or mass but is always statistically dimensionless, as are all the other basic statistical quantities.

While the interval distribution Eq. (1.14) contains the rate a in events per unit time, it always occurs in the product at or $a\,dt$, which is again dimensionless.

Problems

1. Prove analytically that the standard deviation is $\sqrt{m}$ for any Poisson distribution.

2. In the interval distribution for single randomly distributed events, show that the standard deviation is just equal to the average interval, that is, $\sigma = 1/a$.

3. In computing the standard deviation σ of a series of observations x_i, the arithmetic often can be greatly simplified by referring the individual readings to some arbitrary value x_0 (usually chosen as a round number near $\bar{x}$). Then if, as usual,

$$\bar{x} = \frac{1}{n}\sum_{1}^{n} x_i \qquad \text{and} \qquad \sigma^2 \simeq \frac{1}{n-1}\sum_{1}^{n}(x_i - \bar{x})^2$$

show that

$$(a)\quad \bar{x} = x_0 + \frac{1}{n}\sum_{1}^{n}(x_i - x_0)$$

$$(b)\quad \sigma^2 \simeq \left[\frac{1}{n-1}\sum_{1}^{n}(x_i - x_0)^2\right] - \left[\frac{n}{n-1}(\bar{x} - x_0)^2\right]$$

4. In successive 30-min intervals, the number of α rays observed on a certain counter are 13, 9, 16, 9, 14, 11, 17, 12, 7, 12, 15.

(*a*) Compute the average rate in α rays per hour.

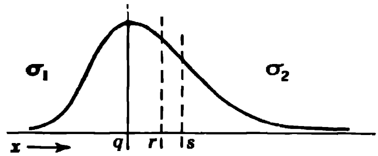
(*b*) If a single additional 30-min observation is made, what are its probable value, standard deviation, and probable error?

(*c*) What is the probable error of the mean value determined in (*a*)?

(*d*) Compare (*c*) with the value expected if the data follow the Poisson distribution.

5. Calculate and plot a Poisson distribution having a mean value of 1.2 for values of x from 0 to 6. What is the standard deviation? What can be said concerning probable error in such an asymmetric distribution?

6. Consider an asymmetric normal distribution having a *modal* (most probable) value of $x = q$, a standard deviation σ_1 for $x < q$, and of σ_2 for $x > q$. The *median* value $x = r$ divides the distribution into two equal areas. The *mean*



value $x = s$ is the average value of x. If the separation $(r - q)$ between the mode and median is small compared with σ_2, show that

$$\text{(median-mode)} = r - q = \sqrt{\frac{\pi}{8}}\,(\sigma_2 - \sigma_1) = 0.627(\sigma_2 - \sigma_1)$$

$$\text{(mean-mode)} = s - q = \sqrt{\frac{2}{\pi}}\,(\sigma_2 - \sigma_1) = 0.798(\sigma_2 - \sigma_1)$$

HINT: Does the assumption $q = 0$ result in any loss of generality?

7. Show that

$$\sum_1^n (x_i - \bar{x})^2 = \sum_1^n x_i^2 - \frac{1}{n}\left(\sum_1^n x_i\right)^2$$

This form is useful when x_i contains one or at most two digits.

3. *Composite Distributions*

Most measurements or calculations in physics involve more than one source of error or of statistical fluctuations. The joint effect of simultaneous but independent sources of statistical fluctuations is now to be considered.

a. Generalized Poisson Distribution. Superposition of Several Independent Random Processes. The complete generalization of the Poisson distribution is usually required in nuclear problems, because several types of radiation will actuate most detection instruments simultaneously. Thus, in ionization-chamber measurements of α rays, there will be present a background composed of α and β rays from radioactive contamination of the walls of the instrument, of cosmic rays, and of γ rays from the earth and the surrounding building materials. If each of several such processes is itself random, the resulting over-all fluctuations may be derived (E24).

Let $x, y, z, \ldots$ be the average number of particles from the several independent random processes, in the time interval chosen. Let them respectively produce specific effects (such as ion pairs) of $a, b, c, \ldots$ per particle. Then the average effect on the instrument is

$$u = ax + by + cz + \cdots \tag{3.1}$$

Generalization of Eq. (2.7) shows (E24) that the square of the S.D. of a single observation of u is given by

$$\sigma^2 = a^2x + b^2y + c^2z + \cdots \tag{3.2}$$

Equations (3.1) and (3.2) are applied to differential measurements by noting that instrumentally subtracted effects correspond simply to negative values of the appropriate coefficient $a, b, c, \ldots$ in Eq. (3.2) and leave the fluctuations unchanged.

In Eqs. (3.1) and (3.2), dimensions may be associated with $a, b, c, \ldots$ and u, but not with $x, y, z, \ldots$. In this case, both the mean value u and the S.D. σ have the dimensions of $a, b, c, \ldots$.

Suppose that a certain ionization chamber receives in unit time an average of $x = 100$ α rays, each producing $a = 10^5$ ion pairs, and also $y = 10^4$ β rays, each producing $b = 10^3$ ion pairs. Then the total average ionization produced is

$$u = 10^5 \times 100 + 10^3 \times 10^4 = 2 \times 10^7 \text{ ion pairs}$$

However, the standard deviation in u is

$$\begin{aligned}\sigma &= \sqrt{(10^5)^2 \times 100 + (10^3)^2 \times 10^4} = \sqrt{10^{12} + 10^{10}} \\ &= \sqrt{1.01 \times 10^{12}} = 1.005 \times 10^6 \text{ ion pairs}\end{aligned}$$

or 5 per cent of u. Thus the α rays produce only half the total ionization but, because of their small number and their large ionization per particle, they account for 99.5 per cent of the statistical fluctuations in the combined ionization effects.

Let this chamber and a second identical ionization chamber be connected in a differential circuit such that an electrometer reads the difference of the ionization in the two chambers. If the second chamber also receives in unit time an average of $z = 100$ α rays, each producing 10^5 ion pairs, then $c = -10^5$ ion pairs because the instrument subtracts cz from $ax + by$. Then the net average differential ionization will be

$$u = 10^5 \times 100 + 10^3 \times 10^4 - 10^5 \times 100 = 10^7 \text{ ion pairs}$$

However, the S.D. in this differential reading is increased to

$$\begin{aligned}\sigma &= \sqrt{(10^5)^2 \times 100 + (10^3)^2 \times 10^4 + (-10^5)^2 \times 100} \\ &= \sqrt{2.01 \times 10^{12}} = 1.41 \times 10^6 \text{ ion pairs}\end{aligned}$$

or 14 per cent of the net u. Note that the differential circuit does not decrease the fluctuations in the total ionization. In fact, *σ has the same value whether the two ionization chambers are connected to oppose each other or to supplement each other.*

In single-particle counting apparatus, such as a Geiger-Müller counter, $a = b = 1$, because the counter discharges once whether the initiating ray is an α ray or a β ray. Thus the $x = 100$ α rays and $y = 10^4$ β rays, if acting in a Geiger-Müller counter, would produce a total of

$$u = 1 \times 10^2 + 1 \times 10^4 = 1.01 \times 10^4$$

counts with a S.D. of

$$\sigma = \sqrt{1 \times 10^2 + 1 \times 10^4} = 1.005 \times 10^2 \text{ counts}$$

or only 1 per cent of u.

We shall return later to Eq. (3.2) for the discussion of the statistics of scaling circuits (Chap. 28).

b. Propagation of Errors. The laws for the propagation of errors are rigorous for the S.D. and can, in fact, be inferred from Eqs. (3.1) and (3.2). In the present section we use probable error merely as a symbol for 0.6745σ, or for $0.6745\sigma_{\bar{x}}$, as required by the context.

Where a physical magnitude is to be obtained from the *summation or* the *differences* of independent observations on two or more physical quantities, the final probable error R of the derived magnitude is obtained from

$$R^2 = r_1^2 + r_2^2 + \cdots \tag{3.3}$$

where $r_1, r_2, \ldots$ are the *absolute* values of the probable errors in the mean values of the several quantities, expressed, of course, in the same units. Thus,

$$(100 \pm 3) + (6 \pm 4) = (106 \pm 5)$$

while

$$(100 \pm 3) - (105 \pm 4) = -(5 \pm 5)$$

The arithmetic of subtraction may be further illustrated by the problem of measuring a counting rate due to some radiation source. A separate measurement must always be made of the natural-background counting rate of the instrument when the source is absent. Suppose that in a time t_b a total of bt_b background counts is recorded. Then the average background rate B and its probable error would be

$$\begin{aligned} B &= (bt_b \pm 0.67\sqrt{bt_b})\frac{1}{t_b} \\ &= b \pm 0.67\sqrt{\frac{b}{t_b}} \end{aligned} \tag{3.4}$$

We note at once that the statistical uncertainty in our evaluation of the background rate depends inversely on the square root of the duration of our observation. Suppose that we now bring a radioactive source near the counter, increasing the true average counting rate to $(S + B)$, where S is due to the source and B to the background. Let $(s + b)$ be the observed counting rate over a period t_s, during which a total of $(s + b)t_s$ counts is recorded. Then our best estimate of $(S + B)$ and its probable error is

$$\begin{aligned} S + B &= [(s + b)t_s \pm 0.67\sqrt{(s + b)t_s}]\frac{1}{t_s} \\ &= s + b \pm 0.67\sqrt{\frac{s}{t_s} + \frac{b}{t_s}} \end{aligned} \tag{3.5}$$

Subtracting Eq. (3.4) from Eq. (3.5) in order to obtain the average counting rate due to the source, we obtain, by Eq. (3.3),

$$S = s \pm 0.67\sqrt{\frac{s}{t_s} + \frac{b}{t_s} + \frac{b}{t_b}} \tag{3.6}$$

It will be noted that the background uncertainty enters twice, once for its fluctuation during the measurement of $(s + b)$ and once for its fluctuation during the measurement of b alone. In the design of counting experiments, it is clear from Eq. (3.6) that the uncertainty in S measured for a fixed time t_s, can always be reduced by prolonging the independent background measurements t_b.

On the other hand, if a physical magnitude Y is to be obtained by *multiplication* or *division* of results of several independent observations on two or more physical magnitudes $y_1, y_2, \ldots$, the *fractional* probable error R/Y in the resulting value of Y depends upon the fractional probable errors $r_1/y_1, r_2/y_2, \ldots$ in the measurement of $y_1, y_2, \ldots$ and is given by

$$\left(\frac{R}{Y}\right)^2 \simeq \left(\frac{r_1}{y_1}\right)^2 + \left(\frac{r_2}{y_2}\right)^2 + \cdots + \left(\frac{r_n}{y_n}\right)^2 \tag{3.7}$$

or its equivalent

$$R \simeq Y\sqrt{\left(\frac{r_1}{y_1}\right)^2 + \left(\frac{r_2}{y_2}\right)^2 + \cdots + \left(\frac{r_n}{y_n}\right)^2} \tag{3.8}$$

Thus we have, for example,

$$(100 \pm 0.3)(6 \pm 0.4) = 600 \pm 600\sqrt{(\tfrac{0.3}{100})^2 + (\tfrac{0.4}{6})^2} = 600 \pm 40$$

$$(100 \pm 3)(100 \pm 4) = 10^4 \pm 10^4\sqrt{(\tfrac{3}{100})^2 + (\tfrac{4}{100})^2} = 10{,}000 \pm 500$$

$$\frac{100 \pm 40}{10 \pm 3} = 10 \pm 10\sqrt{(\tfrac{40}{100})^2 + (\tfrac{3}{10})^2} = 10 \pm 5$$

Equation (3.7) is a good approximation if the fractional errors are small, that is, if $n(r_i/y_i)^2 \ll 1$, as often happens.

c. Significance of the Difference of Two Means. Especially in nuclear physics, many experiments have to be statistically designed as optimum compromises between maximum resolution and maximum intensity. It often happens that the statistical fluctuations in the natural background of a detecting instrument may be comparable with the average value of some feeble radiation effect which is to be measured. In such cases, special care must be taken in interpreting the results of the measurements, and standard tests of the "significance of the difference of means" may need to be applied to the data.

Let m_u and m_v be the true mean values of two independent populations of normally distributed data, while $(\bar{u} \pm \sigma_{\bar{u}})$ and $(\bar{v} \pm \sigma_{\bar{v}})$ are measured values of samples from the two populations, each measurement being based on a sufficient number of observations so that the uncertainty in $\sigma_{\bar{u}}$ and in $\sigma_{\bar{v}}$ is small. Then our best estimate of the difference, $(m_u - m_v)$, of the two means is

$$\begin{aligned} m_u - m_v &\simeq (\bar{u} \pm \sigma_{\bar{u}}) - (\bar{v} \pm \sigma_{\bar{v}}) \\ &= (\bar{u} - \bar{v}) \pm \sqrt{\sigma_{\bar{u}}^2 + \sigma_{\bar{v}}^2} \end{aligned} \tag{3.9}$$

It can be shown† that $(\bar{u} - \bar{v})$ is normally distributed about the true

† See P. G. Hoel, "Introduction to Mathematical Statistics," p. 109, John Wiley & Sons, Inc., New York, 1954, or S. S. Wilks, "Mathematical Statistics," p. 98, Princeton University Press, Princeton, N.J., 1943, on the problems of significance.

mean value $(m_u - m_v)$ with a standard deviation of

$$\sigma_{(\bar{u}-\bar{v})} = \sqrt{\sigma_{\bar{u}}^2 + \sigma_{\bar{v}}^2} \tag{3.10}$$

If, for example, the true mean value is $(m_u - m_v) = 0$, then from Fig. 1.2 there is about a 32 per cent chance that the absolute value of $(\bar{u} - \bar{v})$ will be numerically greater than the standard deviation of its own measurement, $\sigma_{(\bar{u}-\bar{v})}$. Similarly, because Fig. 1.2 shows $P_u = 0.045$ for $u = 2\sigma$, there is only about a 5 per cent chance that the observed absolute value of $(\bar{u} - \bar{v})$ would exceed $2\sigma_{(\bar{u}-\bar{v})}$ if $(m_u - m_v) = 0$.

It is customary but arbitrary in the theory of errors to reject any hypothesis which falls below a "*significance level*" of 5 per cent. Thus, the hypothesis being tested is usually rejected if it predicts that the observation made was so unusual that it should occur less than 5 per cent of the time. Accordingly, an observation of a difference of at least twice the S.D. (or three times the probable error) between two mean values would be said to be "*significant*" and would lead to rejection of any tentative hypothesis that the two true mean values were identical.

For example, suppose that a radiation-safety monitor is searching for β-ray contamination, using an ionization chamber whose natural background has an average value of 10 α rays (10^5 ion pairs per α ray) plus 100 β rays (10^3 ion pairs per β ray) per minute. What is the minimum number of additional β rays per minute which can just be detected in a 30-sec inspection, using the conventional significance level of 5 per cent? From Eq. (3.1), the average background ionization per 30-sec interval is

$$u = ax + by = 10^5 \times 5 + 10^3 \times 50 = 5.5 \times 10^5 \text{ ion pairs}$$

while the S.D. of u is, by Eq. (3.2),

$$\sigma = \sqrt{a^2x + b^2y} = \sqrt{(10^5)^2 \times 5 + (10^3)^2 \times 50} = 2.24 \times 10^5 \text{ ion pairs}$$

Making the valid and simplifying assumption that the additional β-ray activity cz which is just detectable will not alter σ appreciably, we require $cz = 2\sigma$ for the 5 per cent significance level. Taking $c = b = 10^3$ ion pairs per β ray, we find that $z = 2\sigma/c = 2 \times 2.24 \times 10^5/10^3 = 450$ β rays in 30 sec, or 900 β rays/min, as the least amount of β-ray activity which can be "detected" in 30 sec with this instrument.

Evidently, instruments designed for the detection of small activities should have small fluctuations in the background. In the example cited, the major portion of the statistical fluctuations is due to the α rays. Another ionization chamber, having no appreciable α-ray background but the same total average background due entirely to 1,100 β rays/min, would have $\sigma = \sqrt{(10^3)^2 \times 550} = 2.34 \times 10^4$ ion pairs for 30-sec readings. Such a chamber could therefore detect an addition of 2σ ion pairs, or 47 β rays in 30 sec, or an average activity of about 100 β rays/min in a 30-sec observation. Although both ionization chambers considered here have the same average background, their "useful sensitivities" to small sources differ by a factor of 9 (!) because of the important effects of fluctuations in the background.

This numerical example illustrates a broad general principle which is too often overlooked in discussions of the relative sensitivity of various types of detecting equipment. A measure of goodness, or of effective relative sensitivity, is the instrument's response to some small standard source, divided not by the average background but by the magnitude of the fluctuations of the background in unit time. The mere ratio of response divided by average background is meaningless.

In principle, a huge background would be perfectly acceptable if it could be absolutely steady in value. It is the inevitable increase in the absolute value of the statistical fluctuations with increasing background which directs instrument designers to seek low backgrounds.

Problems

1. Two measured quantities and their standard errors are $a = 50 \pm 4$ and $b = 30 \pm 3$. Find the values, with standard error, of the quantities ab, a/b, $(a - b)$, $(a + b)$.

2. A counter has a background of 90 counts per minute as determined from a 1-hr observation. A small sample, tentatively thought to be nonradioactive, is placed near the counter for 5 min. During this time 475 counts are recorded.

(*a*) On a basis of this evidence, is the sample radioactive?

(*b*) If in a period of 20 min 1,900 counts were recorded with the sample present, would it be judged as radioactive?

3. Using a counter having a very accurately measured average background of 120 counts per minute, what must be the duration of an observation of a radioactive source having a constant average activity of about 240 counts per minute if the activity of the source is to be measured with a standard error of 2 per cent?

4. The rate of emission of β rays from a single radioactive substance, for example, P^{32}, is being observed by counting the particles emitted during accurately measured time intervals of equal duration t. The background of the counter is first observed for a time t and is 3,000 counts. Then the source is brought up, and the counting rate rises to 7,000 counts in a time t.

(*a*) From these two observations alone, what is the fractional standard error (in per cent) of the observed counting rate due to the β rays?

(*b*) Why must t be much shorter than the half-period of the radioactive substance for the calculation in (*a*) to be valid?

5. The radioactivity of a long-lived substance emitting β rays is to be measured, using a Geiger-Müller counter. The background of the counter is such that a total of 3,200 counts are recorded in a total running time of t_b min. With the source in position, a total of 3,200 counts are recorded in t_s min.

(*a*) Show that the per cent standard error in the measurement of the source strength, in terms of the observed quantities t_b and t_s, is

$$\left(\frac{100}{\sqrt{3,200}}\right)\left[\frac{\sqrt{t_b^2 + t_s^2}}{(t_b - t_s)}\right]$$

(*b*) What is the per cent standard error if $t_b/t_s = 2$?

(*c*) What is the per cent standard error if $t_b/t_s = 10$?

6. Two Geiger-Müller counters are exposed to the same radiation to determine whether they have the same absolute sensitivity.

(*a*) In the first trial, counter 1 gives a total of 900 counts in the same time counter 2 gives 940 counts. Can this be considered a "statistically significant" difference?

(*b*) If counter 2 gave 990 counts instead of the 940, would this be a "statistically significant" difference?

7. In successive 10-min intervals, the background of a counter is 1,020; 970; 990; 1,040; 950; 1,010; and 980. A radioactive source of long half-period is brought up to the counter, and the increased counting rate, for successive 10-min intervals, is 3,060; 3,100; 2,980; 3,010; 2,950; 3,030. Calculate the average values and standard errors for (*a*) the background, (*b*) the background and source, and (*c*) the source alone.

8. A time T is available in which to measure the counting rate s due to a radiation source, using an instrument whose background counting rate b is not known accurately and must be measured during part of T. Show that maximum accuracy is obtained in the measurement of s by using a time αT for observing the source, and $(1 - \alpha)T$ for observing the background, where

$$\text{"background time"} = 1 - \alpha = \frac{1}{1 + \sqrt{(b + s)/b}}$$

Plot α vs. log (b/s) for $0.01 \leq (b/s) \leq 10$. What is the limiting value of α for very weak sources? For very strong sources?

9. The background b of a counter is to be measured and then the counter is to be used to measure the activity s of a source, all in a fixed time T. If the true mean values are $b = 30$ counts per minute (cpm) and $s = 300$ counts per minute, and if $T = 20$ min, what is the standard error of s in counts per minute when T is divided between background and source measurements such that (*a*) the same total number of counts are recorded for background as with the source in position, (*b*) one-half the time available is used for background, and (*c*) the optimum division of time is utilized? *Ans.*: (*a*) 14 counts per minute; (*b*) 6.0 counts per minute; (*c*) 5.3 counts per minute.

10. Measurements are made with a γ-ray counter on a source of substantially constant average activity.

(*a*) A total count (source plus background) of 8,000 is observed in 10 min. Then, with the source removed, 10 min gave a total of 2,000 background counts. What is the average source strength in counts per minute? What is the standard deviation in this value?

(*b*) If the total time to make measurements is fixed, what is the optimum fraction of time to spend measuring background in part (*a*)?

11. A choice is to be made between two somewhat similar α-ray counters. One is distinguished especially by its low background, the other by its high efficiency.

(*a*) If the average background of a counter is B counts per hour, and the calibration constant or "sensitivity" is S counts per hour per micromicrocurie of, say, radon, show that the fractional standard deviation in the measurement of A $\mu\mu c$ in a time T is

$$\frac{\sigma}{SA} = \sqrt{\frac{SA + B}{TS^2A^2}}$$

(*b*) What is the fractional standard deviation for very weak sources ($A \rightarrow 0$)? For very strong sources ($A \rightarrow \infty$)?

(*c*) The two instruments which are available have $B_1 = 10$, $S_1 = 100$ and $B_2 = 150$, $S_2 = 250$. For very weak sources, should the instrument with the low background or the one with the high sensitivity be used? Which instrument is preferable for strong sources?

(*d*) What is the particular source strength A_0, in micromicrocuries, for which these two instruments give the same fractional statistical error of measurement in any fixed time T?

12. A large group of atoms, whose number is exactly N at $t = 0$, undergoes radioactive decay with decay constant λ and mean life $\tau = 1/\lambda$.

(*a*) State the probability that a given atom has survived at time t.

(*b*) State the probability that a given atom has decayed between $t = 0$ and $t = t$.

(*c*) What is the expectation value $\bar{n}$ of the number of survivors at time t (i.e., the mean number of survivors for many such groups of N similar atoms)?

(*d*) If $t = \tau$, which of the distribution laws studied describe(s) the fluctuation of the number of survivors n about $\bar{n}$?

(*e*) What is the standard deviation of n about $\bar{n}$?

(*f*) What is the probability that a given atom will survive through τ and decay between τ and $\tau + \Delta t$?

(*g*) What is the expectation value $\overline{\Delta n}$ of the number of atoms decaying between τ and $\tau + \Delta t$?

(*h*) If $\Delta t = \frac{1}{10}\tau$, which of the distribution laws studied describe(s) the fluctuation about $\overline{\Delta n}$ of the number of atoms, Δn, decaying between τ and $\tau + \Delta t$?

(*i*) What is the standard deviation of Δn about $\overline{\Delta n}$?

(*j*) At $t = 0$ we have 100 groups of N atoms, each of the above type, which we shall call A, and 100 groups of N atoms, each of a second type B. Observations between τ and $\tau + \Delta t$ result in the following

Type of atom	Mean Δn	S.D. of Δn about mean
A	$\overline{\Delta n_A}$	σ_A
B	$\overline{\Delta n_B}$	σ_B

If $\delta = |\overline{\Delta n_A} - \overline{\Delta n_B}|$, how large a value may δ have without seriously upsetting the hypothesis that types A and B are actually the same atoms?

13. The radium content of an unknown sample is to be determined on an absolute basis by comparison with the γ-ray activity of a radium standard. If A is the observed activity of the unknown and B is the observed activity of the radium standard, then the best value of the ratio A/B is the quantity sought from the measurements. A standardized technique is used, such that each individual measurement of A or of B has a fractional standard deviation of 0.5 per cent.

(*a*) If only one measurement of A and one of B are made, what is the fractional standard deviation of A/B?

(*b*) If three measurements of A are made, what is the fractional S.D. of the average activity $\bar{A}$ of the sample?

(*c*) If three measurements of A and n measurements of B are made, what is the fractional S.D. in the average ratio $\bar{A}/\bar{B}$?

(*d*) It is desired to make enough measurements n on the standard so that no appreciable statistical error is introduced in the final ratio $\bar{A}/\bar{B}$ by uncertainty in the activity $\bar{B}$ of the standard. Again, three measurements are made on A. What is the minimum number n of measurements of B such that the fractional S.D. in $\bar{A}/\bar{B}$ will not exceed 1.2 times the fractional S.D. in $\bar{A}$?

CHAPTER 27

Statistical Tests for Goodness of Fit

Two statistical tests for "goodness of fit" between data and hypothesis will be discussed. It has long been clear that all individual nuclear processes are random in character and hence obey Poisson's distribution and the interval distribution based on it. Data illustrating this fact will be given in Sec. 4 as examples of the application of statistical principles.

1. *Lexis' Divergence Coefficient*

In 1877 the German economist Lexis introduced a divergence coefficient Q^2, defined as the ratio of the average of the squares of the deviations to the arithmetical mean or, in the nomenclature of Chap. 26,

$$Q^2 = \frac{\sum_{1}^{n} (x_i - \bar{x})^2}{n\bar{x}} \tag{1.1}$$

Comparison of Eq. (1.1) with Eq. (2.9) of Chap. 26 shows that

$$Q^2 = \frac{n-1}{n} \frac{\sigma^2}{\bar{x}} \tag{1.2}$$

which, for the normal distribution, can have any value since σ is a parameter of the normal distribution.

For the Poisson distribution, however, Q^2 has a definite value because σ is expressible in terms of the mean value m. Thus, if the data were in perfect agreement with *Poisson's distribution*, we should have, by Eq. (2.7) of Chap. 26,

$$Q^2 = \frac{\sum_{1}^{n} (x_i - m)^2}{nm} = \frac{\sigma^2}{m} = 1 \tag{1.3}$$

We can then compute Q^2 from our data, using Eq. (1.1), and if the observed Q^2 is close to unity we may say that the data seem to follow the Poisson distribution.

While this is very helpful, it is by no means enough, for we need to know how different from unity Q^2 may be expected to be before we should question the randomicity of the process. Such a quantitative test is offered by Pearson's chi-square test.

2. *Pearson's Chi-square Test*

Pearson's (P10) chi-square test determines the probability P that a repetition of the observations would show *greater* deviations from the frequency distribution which is assumed to govern the data. While derived on a basis of the normal distribution, it is successfully used on Poisson and interval distributions because, as stated earlier, the frequency curve of the means of samples drawn from a nonnormal infinite parent population of data is usually more nearly normal than the original population. Moreover, a parent Poisson distribution in which $m \gg 1$ approaches the normal form, as was seen in Figs. 1.1 and 1.3 of Chap. 26.

Whereas the chi-square test provides one of the most decisive statistical criteria, it is too seldom used by physicists, partly because of its unfamiliarity and partly because a large amount of data is required for its most useful applications. Its use should be encouraged.

Pearson's chi-square test may be most simply stated as follows: We define the quantity

$$\chi^2 = \sum^{\iota} \frac{[(\text{observed value})_\iota - (\text{expected value})_\iota]^2}{(\text{expected value})_\iota} \tag{2.1}$$

where the summation is over the total number of independent classifications ι in which the data have been grouped. The "expected values" are computed from any a priori assumed frequency distribution, e.g., normal, Poisson, interval, etc. In general, the data should be subdivided into at least five classifications, each containing at least five events. Secondly, we determine the number of *degrees of freedom F*, which is the number of independent classifications in which the observed series of data may differ from the hypothetical. Then enter Fig. 2.1 and from the values of χ^2 and F determine P, which is the probability that χ^2 should exceed its observed value. Put differently, *P is the probability that, on repeating the series of measurements, larger deviations from the expected values would be observed.*

In interpreting the value of P so obtained, we may say that, if P lies between 0.1 and 0.9, the assumed distribution very probably corresponds to the observed one, while if P is less than 0.02 or more than 0.98 the assumed distribution is extremely unlikely and is to be questioned seriously.

The practical uses of the chi-square test will be illustrated by numerical examples in Sec. 4 below.

For values of $F > 29$, which are neither shown in Fig. 2.1 nor given in the usual chi-square-test tables, it is sufficient to assume that $\sqrt{2\chi^2}$ has a normal distribution with unit standard deviation about a mean

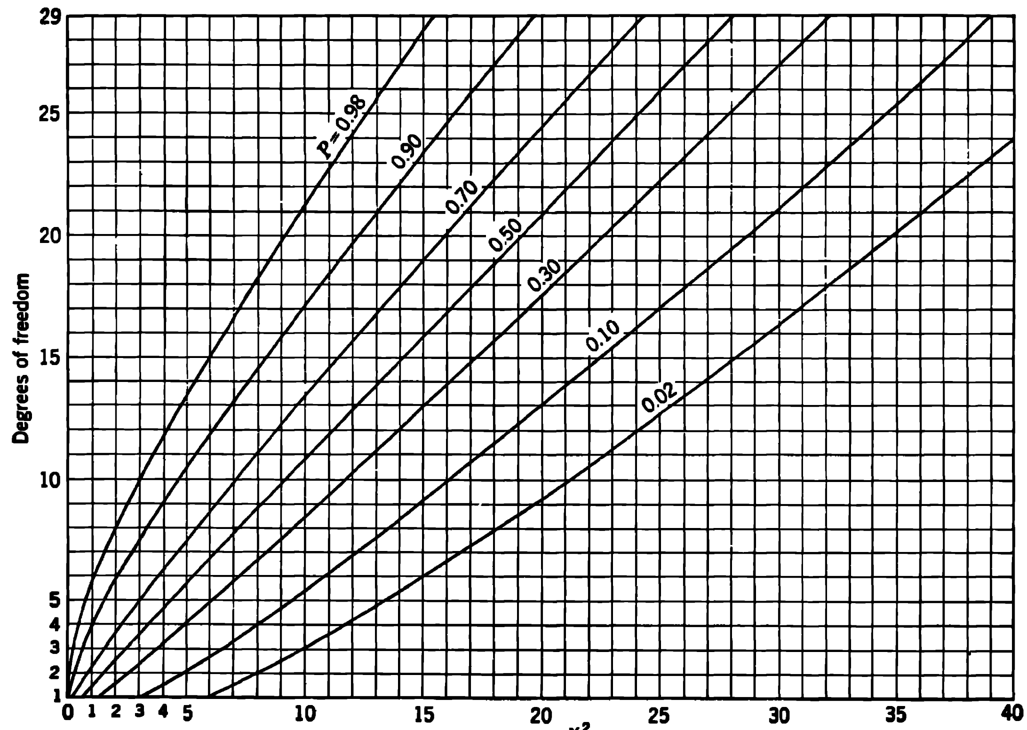


Fig. 2.1 The chi-square test.

value of $\sqrt{2F - 1}$. Then Fig. 1.2 of Chap. 26 may be used for such chi-square tests. In actual statistical practice F seldom exceeds 30 and is usually less than 12.

The chi-square test may be used to determine the validity of any proposed distribution law. Whereas statisticians have devised a number of tests for goodness of fit, physicists should find the chi-square test the most useful of these.

3. *An Extension of the Chi-square Test*

As expressed by Eq. (2.1), χ^2 measures the square of the observed deviations from some assumed frequency distribution. If we fail to specify the distribution assumed but do assume that in a series of n observations of a process, over equal time intervals, the expected value in each interval is constant and equal to the mean value $\bar{x}$ for all the intervals studied, then we would write

$$\chi^2 = \sum_1^n \frac{(x - \bar{x})^2}{\bar{x}} = nQ^2 \tag{3.1}$$

where n values of x are observed and Q^2 is Lexis' divergence coefficient, Eq. (1.1).

It must be emphasized that in Eq. (3.1) we have not yet assumed what distribution governs the observed process; in fact, we have assumed the expected value to be constant. But if the process follows the Poisson distribution, we have seen in Eq. (1.3) that $Q^2 = 1$; hence $\chi^2 = n$. Here the number of degrees of freedom is $F = n - 1$, because our only restriction on the n independent expected values is that they each be equal to $\bar{x}$. It will be seen from Fig. 2.1 that if $n > 5$ then P is between 0.3 and 0.4 for Poisson data so treated. This differs from 0.5 only slightly more than corresponds to the asymmetry of the Poisson distribution and is one justification for a type of application of the chi-square test which is often made on small samples. This test tells far more than can be learned from other readily applied statistical tests.

4. *Examples of Random Fluctuations*

In this section, numerical examples will be given to serve the double purpose of elucidating the application of the statistical principles given in preceding sections and to establish the random character of certain nuclear processes.

a. The Emission of α Rays by Polonium. All modern theories of radioactive decay involve the assumption that in an assembly of nuclei of a given type, e.g., polonium, all the nuclei are identical, independent, and that they each have a definite and constant probability of decaying in unit time (Chap. 15). Since these conditions are precisely the same as the necessary and sufficient conditions for Poisson's distribution, it becomes of fundamental importance to compare the observed statistical fluctuations in the emission of radioactive radiations, from a source of essentially constant strength, with the predictions of Poisson's distribution. The agreement which is found is illustrated by the following example. This is one of the experimental justifications on which many nuclear considerations rest.

To test adequately the random time of emission of α rays from a radioactive substance, a solid angle of 4π should be used so that every emitted ray can be counted, regardless of its direction of emission. This has been experimentally inconvenient but has been done in a few unpublished experiments by Constable and Pollard. Feather (F10) used a 2π solid angle and reported the interval distribution valid for his scintillation study of some 10,000 α rays.

It has been established adequately that at least in noncrystalline sources α rays are ejected uniformly in all directions. This and experimental convenience are the justification for the use of a small solid angle, and such experiments have been made by several workers. Of these, we shall discuss the data of Curtiss (C65), who employed a Geiger point counter to record the α rays emitted within a small solid angle from a polonium source. Between 20,000 and 30,000 α rays were counted in each of eighteen 6- to 7-hr runs extending over an elapsed time of 42 days. Lexis' divergence coefficient was computed for each of the 18 sets of data and showed a gradual approach to $Q^2 = 1.0$ as the source aged and as loose molecular aggregates were detached from the source by the recoil from the disintegration of one of the atoms in the aggregate. After all easily detached aggregates had been torn off, the source more exactly approximated one of constant strength, and the observed emission of α rays approached randomicity.

The resolving time of the counter was sufficiently small that the very short intervals could be faithfully observed. Finite resolving time always tends to lower Q^2 and χ^2 by ignoring short intervals, thus artificially reducing the true dispersion of the counts.

We now consider one of the 18 runs in detail. The number of α rays observed per time interval is recorded for $n = 3{,}455$ equal time intervals. Table 4.1 shows the number of these intervals l_x in which x α rays were observed. Thus no α rays were observed in eight of the intervals, one α ray in 59 intervals, etc., and in all, $\Sigma x l_x = 20{,}305$ α rays were observed. The average number of α rays per interval is thus

$$m \simeq \bar{x} = \frac{20{,}305}{3{,}455} = 5.88 \tag{4.1}$$

Knowing the total number of intervals and the average number of rays per interval, we can now assume that the Poisson distribution may describe the distribution of counts, and compare its predictions with observations. The number of intervals L_x in which x particles would be expected is given by the Poisson formula, Eq. (1.7) of Chap. 26,

$$L_x = nP_x = n\,\frac{m^x}{x!}\,e^{-m} \tag{4.2}$$

Hence, substituting $x = 0$, we expect $L_0 = 3{,}455e^{-5.88} = 9.7$ intervals with no α rays. Table 4.1 shows the calculated values for $x = 0$ to 15 particles per interval. Inspection of the table, or of Fig. 4.1, in which L_x and l_x are plotted against x, shows qualitatively that the observations are in reasonable agreement with Poisson's distribution governing random processes.

TABLE 4.1. CURTISS'S α-RAY DATA

x	l_x (obs)	L_x (calc)	$(l_x - L_x)$		$(l_x - L_x)^2/L_x$
0	8	9.7	−1.7		0.298
1	59	56.9	2.1		0.078
2	177	167.3	9.7		0.562
3	311	327.7	−16.7		0.853
4	492	481.4	10.6		0.233
5	528	565.8	−37.8		2.525
6	601	554.3	46.7		3.930
7	467	465.3	1.7		0.006
8	331	341.8	−10.8		0.342
9	220	223.2	−3.2		0.046
10	121	131.2	−10.2		0.793
11	85	70.1	14.9		3.170
12	24	34.3	−10.3		3.095
13	22	15.5	6.5		2.723
14	6	6.5	−0.5		0.038
15	3	2.6	+0.4	1.0	0.250
≥16	0	1.4	−1.4		
	$n = 3{,}455$				$\chi^2 = 18.942$

Degrees of freedom = 16 − 2 = 14 $\quad\therefore P = 0.2$

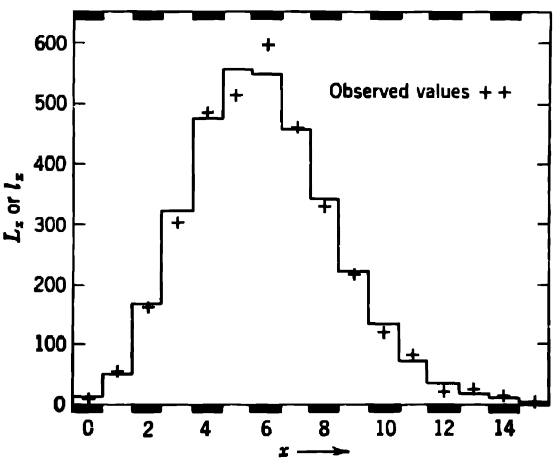


Fig. 4.1 Curtiss's data on the randomicity of emission of α rays from polonium. Observed values (l_x), and the theoretical Poisson distribution histogram (L_x in Table 4.1), are shown for the number of intervals containing x counts, about a mean value of 5.88 counts per interval, and for a total of 3,455 intervals.

To determine the degree of agreement quantitatively, we apply the chi-square test to the data. To compute χ^2 we employ Eq. (2.1), which now takes the form

$$\chi^2 = \sum_x \frac{(l_x - L_x)^2}{L_x} \tag{4.3}$$

Table 4.1 summarizes the calculation leading to $\chi^2 = 18.9$.

The data have been divided into 16 classifications corresponding to $x = 0$ to 15; hence there are originally 16 independent ways in which the observations may be different $(l_x - L_x)$ from the calculations. However, there are not 16 degrees of freedom because two restrictions are placed on these 16 differences. First,

$$\Sigma l_x = \Sigma L_x$$

and secondly,

$$\Sigma x l_x = \Sigma x L_x$$

that is, we have used up two degrees of freedom by specifying (1) the total number of intervals and (2) the total number of events. The second restriction is, of course, equivalent to specifying the average number of rays per interval. There remain, then, $F = 16 - 2 = 14$ degrees of freedom. Entering Fig. 2.1 with these values of χ^2 and F, we find $P = 0.2$, i.e., in 2 cases out of 10 the deviations from Poisson's distribution would be expected to be greater than those here observed. The chi-square test thus gives us quantitative confidence in the randomicity of the process studied.

Studies of statistical theory and applications to cases of radioactive decay have been made by many workers, all tending to substantiate the view that the law of radioactive decay is a statistical law (K31). Kovarik (K43) showed that the β rays from a radium D + E + F mixture follow the Poisson distribution. In fact, all *independent* nuclear processes seem to follow the Poisson and the interval frequency distributions. This does not include some cases of series disintegration, as will be discussed in Chap. 28.

b. Distribution in Time of Cosmic-ray Bursts. As an example of the application of the chi-square test to the interval distribution, we consider the distribution of the time intervals between successive cosmic-ray bursts, as observed by the Montgomerys (M53).

The time of occurrence of 213 bursts in a total of 30.8 hr was observed. Equation (1.15) of Chap. 26 for the distribution of time intervals between randomly spaced events is assumed as a working hypothesis. The chi-square test is then applied to see how closely the observed time intervals between bursts agree with the assumption of random distribution in time. The results are summarized in Table 4.2. In analyzing the data, arbitrary choice is made of the range of time intervals. These are shown in the second and third columns. Thus if two bursts were separated by a time interval of 30 sec, this event would be one of 22 observed entries in the first row. The nomenclature used is analogous to that employed in Table 4.1. Thus l_x denotes the observed values and L_x the values calculated from Eq. (1.15) of Chap. 26, making use of the arbitrarily chosen time intervals, t_1 to t_2, and, from the data (1) the average rate of appearance of bursts (= 1/average interval between bursts) and (2) the total number of observed bursts. Thus the number of degrees of freedom is two less than the number of patterns, or "classifications," studied.

The distribution curve of χ^2 (Fig. 2.1) is to be regarded as only an approximation to the true distribution if the number of independent classifications of data and the minimum number of events per classification are small. Experience and theoretical studies show that *the approximation is usually satisfactory if there are at least five classifications, each containing at least five events.* If there are less than five classifications, each should contain appreciably more than five events. *It is best to combine classifications containing less than five events with an adjacent classification.* Hence in Table 4.2 the intervals between 2,000 sec and

TABLE 4.2. DISTRIBUTION OF COSMIC-RAY BURSTS IN TIME (M53)

Classification	t_1, sec	t_2, sec	l_x (obs)	L_x (calc)	$(l_x - L_x)$	$(l_x - L_x)^2/L_x$
1	0	50	22	19.4	+2.6	0.35
2	50	100	17	17.9	−0.9	0.05
3	100	200	26	30.5	−4.5	0.66
4	200	500	68	63.5	+4.5	0.32
5	500	1,000	47	50.5	−3 5	0.24
6	1,000	2,000	31 }			
7	2,000	5,000	2 }	31.2	+1.8	0.10
8	5,000	∞	0 }			
Total			213	213.0	0	$\chi^2 = 1.72$

Average interval between bursts = 521.6 sec
Degrees of freedom = 6 − 2 = 4
$P = 0.8$

infinity are to be combined with those between 1,000 and 2,000 sec. There are therefore six classifications, or patterns, studied and $6 - 2 = 4$ degrees of freedom.

Entering Fig. 2.1 for Pearson's chi-square test, we find $P = 0.8$, i.e., in 8 out of 10 similar experiments, the deviations from the interval distribution (which rests on the Poisson distribution) would be greater than here observed. There is therefore strong support for the conclusion that the observed phenomena obey the interval distribution as proposed by Eq. (1.15) of Chap. 26, which describes a random process.

c. Randomicity of Geiger-Müller Counter Data. Tables 4.3 and 4.4 snow data taken on two Geiger-Müller counters used in radioactivity measurements. The values of x are the number of impulses per 5-min interval and are due principally to local γ rays and cosmic rays actuating the instrument. We wish to determine, from the spread of these data, whether or not the counter is operating satisfactorily. Abundant evidence exists to show that these counts should be randomly distributed in time. If they are not randomly distributed but tend to show periodicities, then we should suspect the counter in question of some anomalous behavior, such as a spurious periodic discharge superimposed on the true random effect of the incident radiation. The same arguments obviously apply to linear amplifiers, proportional counters, scintillation counters, and all similar detection instruments.

The routine statistical appraisal of these data is shown in the lower half of each table. In Table 4.3 it will be noted that the actual S.D. is slightly less than $\sqrt{\bar{x}}$, suggesting that the dispersion among the data is slightly subnormal. Two of the seven measurements fall outside $(\bar{x} \pm \sigma)$, which is about the correct proportion. The result of the experiment and

TABLE 4.3. ANALYSIS OF GEIGER-MÜLLER COUNTER DATA

Test	x	$x - \bar{x}$	$(x - \bar{x})^2$
1	209	−18	324
2	217	−10	100
3	248	21	441
4	235	8	64
5	224	−3	9
6	223	−4	16
7	233	6	36
Total	1,589	0	990

Eq. (2.1), Chap. 26: $\bar{x} = \frac{1{,}589}{7} = 227$

Eq. (2.9), Chap. 26: $\sigma = \sqrt{\frac{990}{6}} = 12.8$ (from residuals)

Eq. (2.7), Chap. 26: $\sigma = \sqrt{227} = 15.1$ (expected)

Eq. (2.13), Chap. 26: $\sigma_{\bar{x}} = \frac{12.8}{\sqrt{7}} = 4.9$ (from residuals)

Eq. (2.17), Chap. 26: $r_{\bar{x}} = 0.6745 \times 4.9 = 3.3$ (from residuals)

Eq. (2.17), Chap. 26: $r_{\bar{x}} = 0.6745 \times \frac{15.1}{\sqrt{7}} = 3.8$ (expected)

Eq. (1.1), Chap. 27: $Q^2 = \frac{990}{7 \times 227} = 0.623$

Eq. (3.1), Chap. 27: $\chi^2 = \frac{990}{227} = 4.37$

$F = 7 - 1 = 6$

$\therefore P = 0.6$

its probable error of measurement would be recorded as 227 ± 4 counts per 5-min interval, or 45.4 ± 0.7 counts per minute. It is noted that Q^2 is not close to the expected value of unity for a Poisson distribution. Because $Q^2 < 1$, it is again evident that the dispersion of the data is subnormal, i.e., that even greater fluctuations should have been expected from a random distribution. But only the χ^2 test gives us definite information on just how well the data fit a random distribution. The only potential degree of freedom used up in the calculation of the expected values is the average rate $\bar{x} = 227$. With $\chi^2 = 4.37$ and $F = 6$, Fig. 2.1 gives $P = 0.6$. Therefore in 6 out of 10 similar tests we could expect fluctuations greater than those here observed. This is a satisfying result and suggests that this counter is behaving properly.

We now consider the data of Table 4.4, which led to the discovery of a faulty instrument. The very low value of the standard deviation computed from the residuals and the low value of Q^2 at once warn that the dispersion of the data is quite subnormal. However, the chi-square test provides us with a definite numerical gage of the improbability of our result. In 99 cases out of 100, we should expect a greater dispersion of data. We conclude that *either* (1) a very unusual observation has been made or (2) the instrument is faulty and devoted to spurious periodic discharges. The cautious experimenter will surely choose the latter

TABLE 4.4. ANALYSIS OF GEIGER-MÜLLER COUNTER DATA

Test	x	$x - \bar{x}$	$(x - \bar{x})^2$
1	242	−2	4
2	241	−3	9
3	249	5	25
4	246	2	4
5	236	−8	64
6	250	6	36
Total	1,464	0	142

Eq. (2.1), Chap. 26: $\bar{x} = \dfrac{1,464}{6} = 244$

Eq. (2.9), Chap. 26: $\sigma = \sqrt{\frac{142}{5}} = 5.3$ (from residuals)

Eq. (2.7), Chap. 26: $\sigma = \sqrt{244} = 15.6$ (expected)

Eq. (1.1), Chap. 27: $Q^2 = \dfrac{142}{6 \times 244} = 0.097$

Eq. (3.1), Chap. 27: $\chi^2 = \frac{142}{244} = 0.58$

$F = 6 - 1 = 5$

$\therefore P = 0.99$

explanation tentatively and will proceed with further examination of the instrument.

It is instructive to reread and contemplate on the observed values x in Tables 4.3 and 4.4. The naïve observer would usually choose the instrument of Table 4.4, because of the self-consistency and reproducibility of its readings. These are false clues. Variability comparable with or even greater than that exhibited in Table 4.3 *must be exhibited* by a reliable instrument operating on a random process.

Problems

1. Among 927 cosmic-ray bursts observed in 1,344 hr the interval between bursts was less than 30 sec in four instances, between 30 and 60 sec in 10 instances, and greater than 60 sec in the remaining 913 instances. [Cairns, *Phys. Rev.*, **47**: 194L, 631L (1935).] Compute the number of expected intervals of these durations if the bursts are randomly distributed in time. Apply Pearson's chi-square test and estimate the probability that greater deviations from randomicity would be observed in a repetition of the experiment.

2. The results of certain Army records, extending over a period of years, give among other things the number of soldiers killed by the kick of horses.

Number of deaths/time interval	*Frequency observed*
0	109
1	65
2	22
3	3
4	1
5	0
6	0

(*a*) What is the mean value of the number of deaths per time interval?

(*b*) What frequencies would you expect for 0, 1, 2, 3, 4, 5, 6 deaths per time interval?

(*c*) What is the probability that, on repeating this "series of measurements," larger deviations from the expected values would be observed?

3. Consider the spatial distribution of "flying-bomb" hits in a region south of London during World War II. For purposes of analysis, the entire region was divided into 576 squares of equal area ($\frac{1}{4}$ km^2 each). In the total region there were 537 hits altogether. The number of squares, l_x, receiving $x = 0, 1, 2, \ldots$ hits was as given in the table [from R. D. Clarke, *J. Inst. Actuaries*, **72**: 481 (1946)]. Many people believed that the points of impact tended to cluster.

No. hits in one square, x	*No. squares receiving* x *hits,* l_x
0	229
1	211
2	93
3	35
4	7
5	0
6	0
7	1
≥ 8	0

(*a*) Analyze the data given, and determine the probability that a purely random distribution of hits would show *better* agreement with the Poisson distribution.

(*b*) In what mathematical ways, if any, does this problem differ from an analysis of the number l_x of 1-min intervals, out of a total of 576 min, which contain $x = 0, 1, 2, \ldots$ nuclear disintegrations when the average rate is

$$\bar{x} = \frac{537}{576} = 0.932$$

disintegrations per minute?

4. A counter detects the radiation from a small solid angle of a source. A statistical analysis shows these data to obey an interval distribution. Why does this *not* definitely indicate that the disintegrations within the source are randomly distributed in time?

5. In successive 15-min intervals the background of a certain counter is 310, 290, 280, 315, 315, 275, 315. A radioactive source, whose half-period is 14 days, is brought up to the counter, and the increased counting rate, for successive 15-min intervals, is 720, 760, 770, 740, 780, 710, 780, 740. Calculate in counts per minute the average value and standard errors for the (*a*) background, (*b*) background plus source, and (*c*) source alone. Show quantitatively whether or not the data on source plus background can safely be considered to be randomly distributed.

CHAPTER 28

Applications of Poisson Statistics to Some Instruments Used in Nuclear Physics

There are many situations in experimental nuclear physics in which the effect of the detection apparatus is to alter or conceal the randomicity which is actually present in the nuclear process being observed. In some cases this alteration of the statistics of the process can be calculated from the laws which describe purely random distributions. We consider now some of the common practical cases.

1. *Effects of the Finite Resolving Time of Counting Instruments*

Every detection instrument used for counting single rays or particles exhibits a characteristic time constant having the nature of a recovery time. After recording one pulse, the counter is unresponsive to successive pulses until a time interval equal to or greater than its resolving time ρ has elapsed.

The interval distribution [Chap. 26, Eq. (1.14)] shows that short intervals are more likely to occur than are long intervals between successive events in a random distribution. If the interval between two true events is shorter than the resolving time ρ, then only the first event will be recorded. Thus there are both a loss of counts and a distortion of the distribution. Very short intervals are missing in the output. The observed distribution will have an average value and a standard deviation which differ from the true values for the primary random process.

a. Counting Losses Due to Finite Resolving Time. Counter systems really do not count the nuclear events, such as β rays, but rather the intervals between such events. Thus *all counting systems are really interval counters.* The conditions under which ionizing events fail to be recorded depend strongly on the characteristics of the detector and of the amplifier and recording system. Two limiting cases, or types, may be identified easily.

Type I ("Paralyzable"). This type is unable to provide a second output pulse unless there is a time interval of at least ρ between two successive true events. During the response time ρ to an initial event, the recovery of the apparatus is further extended for an additional time ρ

by any additional true events which occur before full recovery has taken place. Thus if five true events are spaced at successive intervals of 2ρ, 0.5ρ, 0.8ρ, 3ρ, only the first, second, and fifth event (corresponding to the first and last intervals) can be recorded. Figure 1.1 illustrates the continued paralysis of the detector until a free interval of at least ρ shall permit relaxation of the apparatus.

Systems of Type I (paralyzable) count only those intervals which are longer than ρ. The interval distribution [Chap. 26, Eq. (1.15)] gives at once the fraction of the intervals which are longer than ρ as $e^{-N\rho}$, where

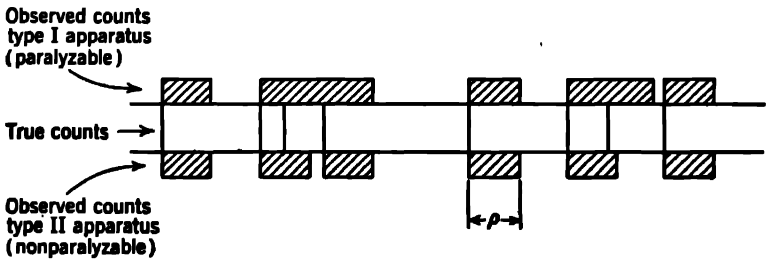


Fig. 1.1 Schematic illustration of the behavior of counting systems having a resolving time ρ. The time axis is from left to right. True counts occur at the times shown by vertical lines along the center section. Apparatus of Type I responds only to intervals longer than ρ. The number of observed counts and the time during which the Type I apparatus is insensitive are shown by shaded blocks. Apparatus of Type II is insensitive for a time ρ after one pulse but then can respond again even if the interval between successive true counts is less than ρ, as illustrated by the triplet. In the hypothetical example shown, there are eight true counts, of which six are recorded by a Type II apparatus and only five are recorded by a Type I apparatus.

N is the average number of true events per unit time. Then if the total number of intervals counted is large compared with unity, the observed counting rate n is simply

$$n = Ne^{-N\rho} \tag{1.1}$$

If the true counting rate N is small enough, only a few intervals (or counts) are missed. Then the observed counting rate is given by the useful approximations

$$n \simeq N(1 - N\rho) \tag{1.2}$$

or

$$N \simeq n(1 + n\rho) \tag{1.3}$$

when $N\rho \ll 1$.

As the true counting rate is increased, differentiation of Eq. (1.1) with respect to N shows that, when $N\rho = 1$, the observed counting rate n passes through a maximum given by

$$n_{max} = \frac{N}{e} = \frac{1}{e\rho} \tag{1.4}$$

It is to be noted that the maximum of the observed counting rate occurs when the average number of true pulses expected per resolving time is unity, that is, $N\rho = 1$. Then $1/e = 0.368$ of the true events is registered. Also the maximum rate of response to *uniformly* spaced pulses (as from an oscillator) would be simply $1/\rho$, which is just e times the maximum observable rate of response of the same apparatus to randomly spaced input pulses.

If the true rate N is increased through values greater than $1/\rho$, then the observed rate n actually decreases, as paralysis of the apparatus becomes increasingly worse because of the scarcity of intervals longer than the resolving time ρ. The behavior is shown in Fig. 1.2.

As the true counting rate approaches infinity, the observed rate will approach zero, i.e., a condition of complete paralysis. Examples of paralyzable apparatus include most forms of electromechanical registers and certain (non-self-quenching) Geiger-Müller counters connected to a conventional high-resistance preamplifier.

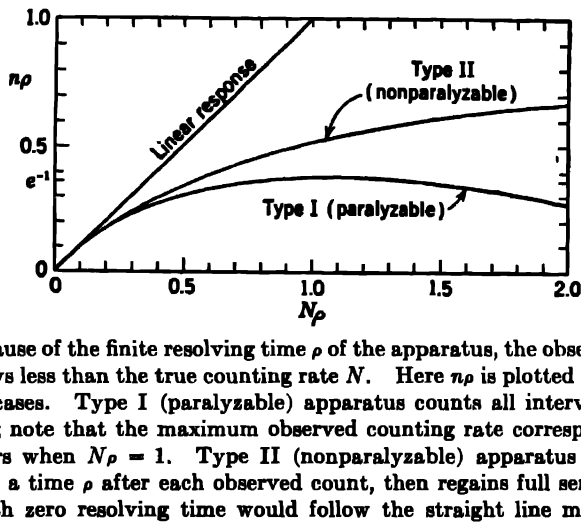


Fig. 1.2 Because of the finite resolving time ρ of the apparatus, the observed counting rate n is always less than the true counting rate N. Here $n\rho$ is plotted against $N\rho$ for two limiting cases. Type I (paralyzable) apparatus counts all intervals which are longer than ρ; note that the maximum observed counting rate corresponds to $n\rho = 1/e$ and occurs when $N\rho = 1$. Type II (nonparalyzable) apparatus is completely insensitive for a time ρ after each observed count, then regains full sensitivity. An apparatus with zero resolving time would follow the straight line marked "linear response."

An excellent generalized statistical treatment of resolving time losses in Type I apparatus for single-channel and coincidence counters, on constant and on decaying sources, has been developed by Schiff (S8).

Type II ("Nonparalyzable"). Thc opposite statistical extreme is found in apparatus which is not affected in any way by events which occur during its recovery time ρ.

Under these circumstances, the apparatus is dead for a time ρ after each recorded event. If the observed counting rate is n, then the fraction of the unit running time during which the apparatus is dead is $n\rho$. The fraction of the time during which the apparatus is sensitive is $1 - n\rho$. This is therefore the fraction of the true number of events which can be recorded, so that

$$\frac{n}{N} = 1 - n\rho \tag{1.5}$$

or

$$N = \frac{n}{1 - n\rho} \tag{1.6}$$

At relatively low counting rates, when $N\rho \ll 1$, we can write Eq. (1.5), to a good approximation, as

$$N \simeq n(1 + n\rho) \tag{1.7}$$

which is the same as for Type I apparatus, Eq. (1.3), provided that $N\rho$, and consequently $n\rho$, is small compared with unity.

Type II apparatus never exhibits complete paralysis. As N is increased, the observed counting rate n rises uniformly, approaching asymptotically the value

$$n_{max} = \frac{1}{\rho} \qquad \text{for } N = \infty \tag{1.8}$$

The general nature of the response curve is shown in Fig. 1.2. To an infinitely strong source of radiation, the apparatus responds periodically, with a frequency of $1/\rho$, all traces of statistical randomicity having been erased.

Equipment of Type II is illustrated by a fixed-gas counter (non-self-quenching) connected to a quenching preamplifier designed to maintain the applied voltage below the counting threshold for a time ρ, during which both the counter and the recording circuit are able to effect complete recovery. The same characteristics would be exhibited by a self-quenching counter connected to a very sensitive preamplifier capable of responding to all pulses received after the dead time of the counter. Most proportional counters and scintillation counters also follow the behavior of Type II.

b. Measurement of Resolving Time in Single-channel Counters. All that has been said above assumes that the resolving time is independent of the counting rate. There is some evidence (M74) that the dead time ρ in self-quenching counters may decrease when the counting rate is elevated to very high values. But at the lower counting rates met in most measurements, the assumption of constant resolving time agrees well with the observations. Many apparatus, however, do not conform perfectly to either of the limiting cases treated above but show resolution characteristics intermediate between Types I and II. Happily, the expressions for both limiting types converge at low counting rates. If $N\rho \leq 0.05$, the exact expressions for n/N differ from each other by less than 0.1 per cent, and the relation

$$N \simeq n(1 + n\rho) \tag{1.9}$$

may be used for any single-channel counting apparatus.

The resolving time of a reasonably well-designed Geiger-Müller counter and amplifier will usually be found to be between 3 and 6×10^{-4} sec, which is equivalent to between 5 and 10×10^{-6} min. Then at an observed counting rate of 1,000 counts per minute, the fraction of the counts which are lost is $n\rho = 5$ to 10×10^{-3}, or 0.5 to 1 per cent. At 2,000 counts per minute, the same apparatus loses 1 to 2 per cent of the counts. Based on Eq. (1.9), counting losses are often cited simply as "*per cent loss per thousand counts per minute.*" Observed counting rates can be corrected easily and accurately for counting losses if $N\rho < 0.05$, i.e., up to observed rates of 5,000 to 10,000 counts per minute for most Geiger-Müller counters.

Scintillation counters, employing anthracene or similar phosphors, can have pulse widths of less than 10^{-7} sec. With suitably fast amplifiers, resolving times of 10^{-6} sec have been realized. Hence counting experiments with such equipment can be carried out accurately at counting rates of the order of 500,000 per minute. Certain proportional counters and linear pulse amplifiers can also have resolving times of the order of a microsecond or less.

One of the simplest satisfactory methods (B26, R12) for measuring the resolving time of single-channel counting apparatus is to compare the response of the apparatus to the radiation from two approximately equal sources, taken separately and then taken simultaneously. Let B be the true average background counting rate when neither source is present, and let N_A and N_B be the true elevation of the counting rate for each of the two sources. Then the observed counting rates for each of the two sources, including background, are n_A and n_B, where

$$N_A + B = n_A(1 + n_A\rho) \tag{1.10}$$
$$N_B + B = n_B(1 + n_B\rho) \tag{1.11}$$

Then when both sources are measured simultaneously, the sum of their radiation will elevate the true rate to $N_S + B = N_A + N_B + B$, but the observed rate will be only n_S, where

$$N_A + N_B + B = n_S(1 + n_S\rho) \tag{1.12}$$

provided that all counting rates are small enough that $n_S\rho \ll 1$. Subtracting Eq. (1.12) from the sum of Eqs. (1.10) and (1.11), and solving for ρ, we have

$$\rho = \frac{n_A + n_B - n_S - B}{n_S^2 - n_A^2 - n_B^2} \tag{1.13}$$

A useful transformation of Eq. (1.13) is obtained by setting

$$\delta = n_A + n_B - n_S - B \tag{1.14}$$

where, physically, δ is the difference between the counting losses in the observation on both sources taken simultaneously and the sum of the counting losses in the two observations on the two sources taken singly. Then Eq. (1.13) becomes

$$\rho = \frac{\delta}{2n_An_B - 2(\delta + B)n_S - (\delta + B)^2} \tag{1.15}$$

or, to an approximation which is usually satisfactory,

$$\rho \simeq \frac{\delta}{2n_An_B} \tag{1.16}$$

In carrying out an estimation of the resolving time ρ by this "two-source method," the observations should be taken in the order n_A, n_S,

n_B, that is, on A, $A + B$, and finally B alone. In this way, the single and the combined readings on each source are obtainable without moving the source between successive readings. This procedure avoids errors due to failure to reproduce the source positions accurately, as can occur when the order n_A, n_B, n_S is used. It is also important that, when the combined radiation of both sources is measured, the two sources should be sufficiently separated from each other so that neither source can scatter any of the radiation from the other source into the counter.

The National Bureau of Standards distributes standard γ-ray sources of certified radium or Co^{60} content, which are convenient for calibrating counting apparatus. One series includes 5 ml of dilute HCl in flame-sealed glass ampoules containing Ra in the accurately graduated amounts 0.1, 0.2, 0.5, 1, 2, 5, 10, etc., μg. Pairs of these ampoules may be used in the two-source method of Eq. (1.13).

Alternatively, a series of such standard sources may be used to obtain a direct plot of observed counting rate n against source strength S. By fitting the best straight line (linear response, as in Fig. 1.2) to the lower end of such a curve, the nonlinearity of response of the counting apparatus can be evaluated empirically without making any assumptions regarding the detailed mechanism by which it loses counts. The slope of the straight line representing linear response N can be adjusted most accurately by noting that the counting loss $(N - n)$ and especially the counting loss per unit source strength, that is, $(N - n)/S$, must both extrapolate to zero at zero net counting rate. Thus the slope of the line of linear response N can be accurately adjusted, by successive approximations (K36), from auxiliary plots of $(N - n)/S$ against source strength S.

In the mathematical treatment, we have assumed thus far that the apparatus has only one controlling time constant, which is independent of counting rate. Certain scaling circuits, connected to slowly operating mechanical registers or recorders, may exhibit two important time constants. At low counting rates, the losses may be determined only by the resolving time of the counter and the first stage of the amplifier, as described in the preceding paragraphs. As the counting rates are elevated, a situation will occur in which, for example, a scale of 4 will occasionally receive five or more pulses in less than the time interval required for action of the mechanical register which it should be driving each time four counts are received. Then the resolving time of the mechanical register also becomes important. Consequently, the mathematical analysis of the scaling losses in such cases is more complicated (L30, L23) and will not be discussed here because usually it can be avoided by proper design of apparatus, e.g., by increasing the scaling factor.

c. Effect of Loss of Short Intervals on the Standard Deviation of the Output of Single-channel Counters. When N is the true average rate of a random process which is observed for a total time T, the "expectation," or average, number of true events is NT. In the preceding sections we have seen that the average, or expected, number of *observed* events nT is smaller than NT by an amount which depends on the resolving time ρ, the average rate N, and the type of apparatus. All the expressions developed there concern the usual nuclear laboratory case in which a large number of events are observed, that is, $NT \gg 1$. A much more complicated analysis must be performed to determine nT when only a few events are observed.

The variance (square of the standard deviation) of the expected true number of events is also NT, by Eq. (2.7) of Chap. 26. However, the variance of the *observed* number of events is not given simply by nT, even when $nT \gg 1$, because the number of short intervals which are lost depends both on the type of apparatus and on $n\rho$ or $N\rho$. Because the more abundant short intervals are removed, the variance of the observed distribution may be markedly smaller than of the parent distribution.

Various approximate or asymptotic solutions for the average number of registrations nT and for the variance σ^2 of the number of registrations have been developed by several workers.

For Type I (paralyzable) apparatus, the expected average number of registrations nT is (F29)

$$nT = (NT - N\rho - 1)e^{-N\rho} + 1 \tag{1.17}$$

which reduces to Eq. (1.1) for the usual experimental case in which $T \gg \rho$. The variance is approximately (K42)

$$\sigma^2 = NT\left(1 - 2N\rho + \frac{N\rho^2}{T}\right) \tag{1.18}$$

which for inferior apparatus arrangements can be (L19) even as small as $NT/2$. Feller (F29) has developed a much more complicated and presumably more exact expression for the variance of paralyzable apparatus.

For Type II (nonparalyzable) apparatus, the asymptotic expansion of the general solutions obtained with the use of operational calculus leads to an approximate expression for the average expected value of (F29)

$$nT \simeq \frac{NT}{1 + N\rho} + \frac{N^2\rho^2}{2(1 + N\rho)^2} \tag{1.19}$$

which reduces to Eq. (1.6) when $N\rho \ll 1$. The variance of the number of registrations is approximately (F29)

$$\sigma^2 \simeq \frac{NT}{(1 + N\rho)^3} \tag{1.20}$$

which is some 20 per cent smaller than the simple Poisson value nT when $N\rho = 0.1$.

d. Random Coincidences in Coincidence and Anticoincidence Circuits. Many types of measurement are made using two (or more) counters exposed to the same source of nuclear radiation. The outputs from the two counters may then be fed through a coincidence circuit, from which an output pulse is delivered only if pulses were received "simultaneously" from the two counters.

For example, a source of Al^{28}, whose decay scheme is shown in Fig. 4.2 of Chap. 3, might be placed between a β-ray counter and a γ-ray counter. Then a *true coincidence* would be registered when the two counters were triggered by the β ray and the prompt γ ray emitted by the same atom. In addition to such true coincidences, there will be false or *random coincidences* which are produced when a β ray and an unrelated γ ray actuate the counters within the resolving time of the apparatus.

Figure 1.3 is an illustrative experimental situation. Channel 1 receives random pulses (e.g., from β rays) at an average true rate N_1 and has a resolving time ρ_1. Channel 2 also receives randomly distributed pulses (for example, γ rays from the same source) at an average true rate N_2 and has a resolving time ρ_2. Let $N_{1,2}$ represent the true coincidence rate, which is generally much smaller than the singles rates N_1 and N_2.

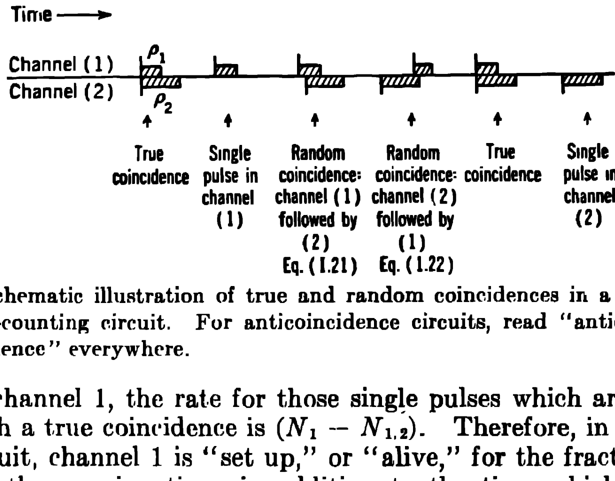


Fig. 1.3 Schematic illustration of true and random coincidences in a two-channel coincidence-counting circuit. For anticoincidence circuits, read "anticoincidence" for "coincidence" everywhere.

Then, in channel 1, the rate for those single pulses which are *not* associated with a true coincidence is $(N_1 - N_{1,2})$. Therefore, in the coincidence circuit, channel 1 is "set up," or "alive," for the fraction $(N_1 - N_{1,2})\rho_1$ of the running time, in addition to the time which it spends responding to true coincidences. In channel 2, single pulses which are not associated with true coincidences are arriving at an average rate $(N_2 - N_{1,2})$. The random-coincidence rate due to single pulses in channel 1 being followed, within its resolving time ρ_1, by single pulses in channel 2 is therefore

$$(N_1 - N_{1,2})\rho_1(N_2 - N_{1,2}) \tag{1.21}$$

To these we must add additional random coincidences due to single pulses in channel 2 which are followed within its resolving time ρ_2 by random single pulses in channel 1. Because channel 2 is alive for the fraction $(N_2 - N_{1,2})\rho_2$ of the running time, aside from its response to true coincidences, this additional random-coincidence rate is

$$(N_2 - N_{1,2})\rho_2(N_1 - N_{1,2}) \tag{1.22}$$

Both these two types of random coincidence are illustrated in Fig. 1.3. "Double random coincidences," such as would be caused by two pulses in channel 1 within the resolving time of channel 2, can be made negligible by keeping $N_1\rho_1 \ll 1$ and $N_2\rho_2 \ll 1$. Then the total random-coincidence rate is

$$\begin{aligned} N_{\text{random}} &= (N_1 - N_{1,2})\rho_1(N_2 - N_{1,2}) + (N_2 - N_{1,2})\rho_2(N_1 - N_{1,2}) \\ &= (N_1 - N_{1,2})(N_2 - N_{1,2})(\rho_1 + \rho_2) \end{aligned} \tag{1.23}$$

In practice, the singles rates N_1 and N_2 are usually large compared with the true coincidence rate $N_{1,2}$. Then Eq. (1.23) becomes approximately

$$N_{\text{random}} \simeq N_1N_2(\rho_1 + \rho_2) \tag{1.24}$$

If N_1 and N_2 are each proportional to source strength, we note from Eq. (1.24) that the random coincidences increase with the square of the source strength. This condition imposes an upper limit on the useful source strength in any coincidence experiment, because at least half the total observed coincidences should be true coincidences.

In anticoincidence circuits, pulses are recorded from channel 1 provided that there is no coincident pulse in channel 2. It can be seen that the number of random anticoincidences is also given by Eq. (1.23).

Problems

1. The decay $N = N_0e^{-\lambda t}$ of a radioactive substance is being observed with a paralyzable counter whose resolving time is ρ. Write an expression for the observed counting rate n as a function of time. Assume that $N_0\rho \sim 0.1$ and no resolving-time corrections are made. Under what conditions will the apparent half-period of radioactive decay be of the order of 10 per cent greater than the true half-period $0.693/\lambda$?

2. The resolving time of a γ-ray counter and amplifier is to be determined. Two radioactive sources A and B are first measured separately and then together. The observed counting rates are n_A for source A, n_B for source B, and n_S for $(A + B)$, each including the small background rate B.

(*a*) Derive an expression for the resolving time ρ of the apparatus, in terms of these three observed counting rates. Assume that n_S is small compared with the reciprocal of the resolving time.

(*b*) Calculate the resolving time of an apparatus if $B = 100$ counts per minute, $n_A = n_B = 4{,}800$ counts per minute, and $n_S = 9{,}120$ counts per minute.

(*c*) What would be the true counting rate for the source A?

Ans.: (*a*) See Eq. (1.15); (*b*) 10.3 μmin; (*c*) 5,050 counts per minute.

3. The resolving time of a γ-ray counter and amplifier is to be determined. Two radioactive sources C and D are available, and D is known to be exactly R times as strong as C. The observed counting rates are n_C counts per minute for source C and n_D for source D, including a background counting rate of B. Assume that n_D is small compared with the reciprocal of the resolving time.

(*a*) Show that the resolving time ρ of the apparatus is given by

$$\rho = \frac{n_CR - n_D - (R - 1)B}{n_D^2 - n_C^2R}$$

(*b*) Calculate the resolving time of an apparatus if $n_C = 3{,}050$ counts per minute, $n_D = 8{,}690$ counts per minute, $B = 100$ counts per minute, and $R = 3.00$.

(*c*) What would be the true counting rate for the source C?

Ans.: (*b*) 5.5 μmin; (*c*) 3.001 counts per minute.

4. In the two-source method for determining the resolving time ρ, show that a close approximation for the standard error $\sigma(\rho)$ in the determination of ρ, in an experiment of total duration $3T$, is

$$\sigma(\rho) \simeq \frac{1}{n_A}\sqrt{\frac{1}{n_A T}}$$

if $n_A \simeq n_B$ and a total time of $3T$ is approximately equally divided between measurement of n_A, n_B, and n_S, the background B being known in advance with negligible error.

5. A certain electromechanical register is found to follow just 120 periodic pulses per second without jamming.

(*a*) What is its maximum counting rate of randomly distributed pulses, as the average rate of random pulses is increased without limit?

(*b*) At the maximum observed rate of counting, what is the true average rate?

6. A Western Electric telephone register is found to have a maximum counting rate of 240 per minute for random pulses. Compute the true counting rate when this register shows 10, 30, 60, 90, 120, 180, 240 counts per minute from a random process. Plot these observed counting rates as abscissas against true rates as ordinates.

7. In the fall of 1947 an amateur long-range weather forecaster set out to predict the times of the snowstorms in Boston for the coming winter. Assuming that winter lasts from December 17 to March 15, a total of 90 days, and that the average number of snowstorms per winter for the last 20 years is 15, he predicted a total of 15 snowstorms and assigned a date to each one at random. Thus he divided the winter into 24-hr intervals and for 15 of these intervals, chosen at random, he predicted snow. It turned out, however, that there were actually 27 snowstorms that winter. Assume that each of these lasted exactly 6 hr and that they were randomly distributed throughout the winter. If we agree to call a prediction successful when some snow (not necessarily the whole amount coming down in a snowstorm) fell during an interval for which snow was predicted, then what is the probable number of successful predictions? *Ans.:* About six.

2. *Scaling Circuits*

In a typical scaling circuit, one output pulse is produced for every s input events. If the events at the input are randomly distributed in time and have an average rate a, the scaling circuit conceals the short intervals, tends to average out the variations of interval length, and produces an approximately periodic output (H67) whose mean frequency is a/s. Those counting losses which are due to the resolving time of electromechanical registers can be made negligibly small by the use of scaling circuits having a sufficiently large scaling factor s. Electronically, two types of scaler are now in common use. The two types are the scale of 2, which is cascaded to give instruments having $s = 2^n = 2, 4, 8, 16, 32, 64, \ldots, 4{,}096, \ldots$, and decade scalers having $s = 10^n = 10, 100, 1{,}000$.

a. Generalized (*s*-fold) Interval Distribution. We shall assume at first that the input pulses delivered to the scaler are randomly distributed in time, at an average rate a, or average interval $1/a$. The length of an interval between output, or "s-fold," pulses may be called an "s-fold interval." The intervals *between* successive input pulses contain zero events; the intervals between successive s-fold pulses contain $s - 1$ events (such as counts from a Geiger-Müller counter or from a scintillation counter).

The Poisson distribution [Chap. 26, Eq. (1.6)] shows that the probability that an s-fold interval of duration t will contain exactly $s - 1$ events is

$$P_{s-1}(t) = \frac{(at)^{s-1}}{(s-1)!} e^{-at} \tag{2.1}$$

The probability of one event occurring in an additional time dt is simply

$$P_1(dt) = a\,dt \tag{2.2}$$

The probability of $s - 1$ events in t, and the sth event between t and $t + dt$, is therefore

$$\begin{aligned} dP_t &= P_{s-1}(t)\,P_1(dt) \\ &= \frac{a^s t^{s-1}}{(s-1)!} e^{-at}\,dt \end{aligned} \tag{2.3}$$

which is the generalized s-fold interval distribution. Equation (1.14) of Chap. 26 is seen to correspond to the special case in which $s = 1$. Equation (2.3) expresses the probability that an s-fold interval will have a duration between t and $t + dt$ when a is the average rate and a/s is the average rate of s-fold counting.

We note that Eq. (2.3) is already normalized, and this can be verified by finding the probability that the s-fold interval will have some duration between zero and infinity. Thus

$$\int_0^\infty dP_t = \frac{a^s}{(s-1)!}\int_0^\infty t^{s-1}e^{-at}\,dt = \frac{a^s}{(s-1)!}\,\frac{(s-1)!}{a^s} = 1 \tag{2.4}$$

The probability P_T that an s-fold interval will be equal to or shorter than a time T can be obtained by integration of Eq. (2.3). Then

$$P_T = \int_0^T dP_t = \frac{a^s}{(s-1)!}\int_0^T t^{s-1}e^{-at}\,dt \tag{2.5}$$

This integral can be evaluated by successive integration by parts, yielding

$$P_T = 1 - e^{-aT} - \frac{aT}{1}e^{-aT} - \frac{(aT)^2}{1 \times 2}e^{-aT} - \cdots - \frac{(aT)^{s-1}}{(s-1)!}e^{-aT} \tag{2.6}$$

Each of these terms is simply the Poisson probability of 0, 1, 2, . . . $(s - 1)$ events in the time T, or

$$P_T = 1 - (P_0 + P_1 + P_2 + \cdots + P_{s-1}) \tag{2.7}$$

or, since $\sum_0^\infty P_i = 1$,

$$P_T = P_s + P_{s+1} + \cdots + P_\infty \tag{2.8}$$

This result could have been written directly by noting that the probability for an s-fold pulse within the time T is simply the Poisson probability of s, *or more*, events in the time T.

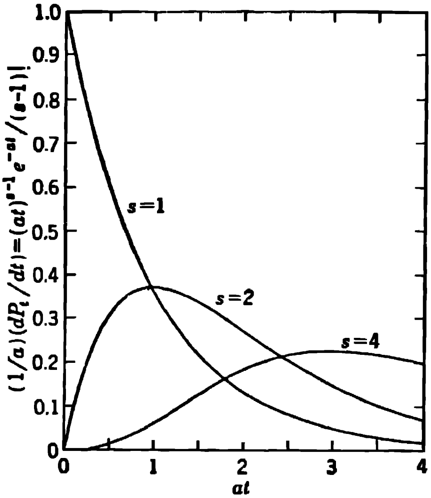


Fig. 2.1 Generalized s-fold interval distribution. Abscissas: average number of randomly distributed events at in time interval t. Ordinates: probability of s-fold interval, of duration between at and $a(t + dt)$. The area under each curve between time zero and at is the probability that an s-fold interval will be shorter than at, Eq. (2.5). Note that the duration at of the most probable s-fold interval is $s - 1$, Eq. (2.10), i.e., the average time for $s - 1$ random events. The average interval, however, is $at = s$, Eq. (2.9). Note the great reduction in the probability of short intervals which is produced even by these very small scaling factors. Because of the analytical form of Eq. (2.3), these curves also represent the Poisson probability of getting $s - 1$ random events in the time t when the average expected number is at, Eq. (2.1), or Eq. (1.6) of Chap. 26.

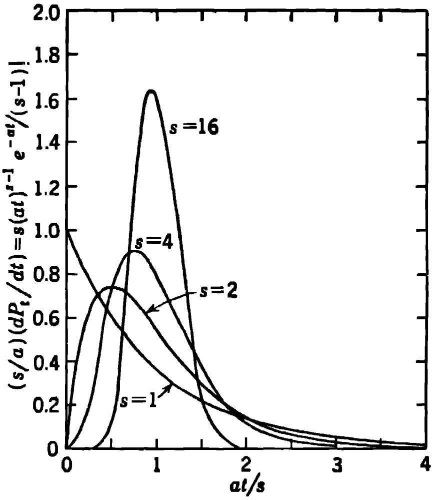


Fig. 2.2 Curves similar to Fig. 2.1 but normalized to a time axis at/s, so that the average s-fold interval is $at/s = 1$. The most probable s-fold interval is $at/s = (s - 1)/s$. The ordinates are adjusted so that the area under each curve is unity. Note the strong regularizing action of the larger scaling factors, e.g., for $s = 16$ the distribution of intervals appears almost normal. The fractional standard deviation of the distribution of s-fold intervals decreases as $\sqrt{1/s}$, as given by Eq. (2.15). Thus the chance that the duration of an s-fold interval will vary greatly from the average s-fold interval decreases markedly as s increases.

Of course, the probability that an s-fold interval will be longer than T is simply $1 - P_T$. Useful numerical tables of P_T have been compiled by Molina (M49).

The distribution of s-fold intervals is shown in Figs. 2.1 and 2.2.

b. Average s-fold Interval. The average s-fold interval has the duration

$$\begin{aligned}\bar{t}_s &= \int_0^\infty t\,dP_t \\ &= \frac{a^s}{(s-1)!}\int_0^\infty t^s e^{-at}\,dt \\ &= \frac{a^s}{(s-1)!}\frac{s!}{a^{s+1}} \\ &= \frac{s}{a}\end{aligned} \tag{2.9}$$

as expected from the fact that the average interval of the parent random process is $1/a$.

c. Most Probable s-fold Interval. The probability that an s-fold interval lies between t and $t + dt$ is given by Eq. (2.3). Then $y = dP_t/dt$ is the differential probability of an interval of duration t, and dy/dt is the variation of this probability with the duration t of the s-fold interval. The most probable duration t_0 occurs when

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{a^s t^{s-1} e^{-at}}{(s-1)!}\right] = 0$$

which is when $t = (s - 1)/a$, thus

$$t_0 = \frac{s-1}{a} \tag{2.10}$$

It will be noted that the most probable interval is slightly shorter than the average interval $\bar{t}$, such that

$$t_0 = \frac{s-1}{s}\bar{t} \tag{2.11}$$

For scale of 1 the most probable interval is zero, while when s is very large the most probable interval approaches the average interval.

d. Standard Deviation of s-fold Intervals. The deletion of short intervals and the approximately periodic output of the scaler is sometimes referred to as the *regularizing action* of a scaler. The variance σ^2 of the s-fold intervals is a measure of this "smoothing effect" and is

$$\sigma^2 = \int_0^\infty (t - \bar{t})^2\,dP_t = \frac{a^s}{(s-1)!}\int_0^\infty \left(t - \frac{s}{a}\right)^2 t^{s-1}e^{-at}\,dt \tag{2.12}$$

which, on expansion and evaluation, leads to

$$\sigma^2 = \frac{s}{a^2} \tag{2.13}$$

or, for the standard deviation σ, we have

$$\sigma = \frac{\sqrt{s}}{a} = \frac{\bar{t}}{\sqrt{s}} \tag{2.14}$$

Then the *fractional* standard deviation of s-fold intervals is

$$\frac{\sigma}{\bar{t}} = \frac{1}{\sqrt{s}} \tag{2.15}$$

Thus the fractional standard deviation in the time $\bar{t}$, required to accumulate s random events, decreases with the inverse square root of the total number of counts, for it does not matter what type of apparatus is used to tally the s events.

This important result shows directly what fluctuations are to be expected in counting observations based on measurements of the time required to accumulate a predetermined number of counts. Thus the fractional standard deviation of the time required to accumulate a total of, for example, $2^{12} = 4{,}096$ counts is simply $\sqrt{1/4{,}096} = \frac{1}{64} = 1.56$ per cent.

It should be especially noted that this is the same as the algebraic form Eq. (2.15) of Chap. 26 for the fractional fluctuation, or error, which would be associated with observations of the number of counts accumulated in a predetermined time.

Scaling circuits may also be understood from the viewpoint of the generalized Poisson distribution of Eqs. (3.1) and (3.2) of Chap. 26. If x is the expectation number of random events, and if the specific effectiveness is $1/s$ per event, then Eq. (3.1) of Chap. 26 shows that the expected average number of s-fold pulses is simply

$$u = \frac{x}{s} \qquad \text{if } u \text{ is integral or if } x \gg s \tag{2.16}$$

while the expected standard deviation σ is given by Eq. (3.2) of Chap. 26 as

$$\sigma^2 = \frac{x}{s^2} \tag{2.17}$$

and the fractional standard deviation of the number of s-fold registrations in a fixed time is

$$\frac{\sigma}{u} = \frac{\sqrt{x/s^2}}{x/s} = \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{su}} \tag{2.18}$$

which is the same as Eq. (2.15) of Chap. 26 and as Eq. (2.15) above, because u can have any integral value and is unity for the case considered in Eq. (2.15) above.

e. Interpolation. Many conventional scaling circuits are provided with interpolation lights or meters. Then if counting is stopped at a predetermined time, the exact number of input counts x is given by

$$x = su + \Delta \tag{2.19}$$

where u is the number of s-fold pulses and Δ is the number of single interpolation pulses. It is to be noted that the interpolation pulses Δ are only statistically significant if they are comparable with or greater than the standard deviation of x. We can ignore the interpolation pulses in many practical cases without introducing a statistically significant error. Suppose that we decide that Δ is to be ignored whenever its maximum possible value, $\Delta_{max} = s - 1$, cannot exceed some arbitrary fraction β of the standard deviation in x. Then we can omit the chore of interpolation whenever

$$\Delta_{max} = s - 1 < \beta \sqrt{su + \Delta} > \beta \sqrt{su} \tag{2.20}$$

Squaring, and solving the second inequality for u, we have

$$u > \frac{(s - 1)^2}{\beta^2 s} \tag{2.21}$$

or, in the practical case of $s \gg 1$, and $\beta \sim \frac{1}{2}$, interpolation is pointless whenever

$$u > 4s \tag{2.22}$$

f. Chi-square for the Output from a Scaler. The output from a scale of s may be tested for fidelity and for randomicity of the input process by computing the standard deviation of the s-fold output from its residuals, Eq. (2.9) of Chap. 26, and comparing with the theoretically expected values of Eq. (2.15) or Eq. (2.18).

A more objective appraisal of the data is obtained by applying the χ^2 test. This can be done by computing the input events, from the s-fold output readings, and then applying Eq. (3.1) of Chap. 27 to the input or scale-of-1 events. More conveniently, one may wish to compute χ^2 for the input process directly from the s-fold output readings. Then two cases arise, depending on whether the series of output observations are made over a predetermined and fixed time interval (variable number of counts) or over a predetermined total number of counts (time variable).

Case I, Constant Time Interval. In a series of n successive equal time intervals, let the number of observed s-fold output pulses be $u_1, u_2, \ldots,$ u_n (u can be nonintegral if interpolation has been resorted to, thus $u = x/s$). Then the average s-fold output is

$$\bar{u} = \frac{1}{n} \sum_{1}^{n} u_i \tag{2.23}$$

and χ^2, of Eq. (3.1) of Chap. 27 but in terms of the s-fold output counts, becomes

$$\chi^2 = \sum_{1}^{n} \frac{(x_i - \bar{x})^2}{\bar{x}} = \sum_{1}^{n} \frac{(su_i - s\bar{u})^2}{s\bar{u}}$$

$$= \frac{s}{\bar{u}} \sum_{1}^{n} (u_i - \bar{u})^2 \tag{2.24}$$

The expected value of the summation in Eq. (2.24) is $n - 1$ times the variance σ^2 of u. If the input process is random, then Eq. (3.2) of Chap. 26 gives $\sigma^2 = \bar{x}/s^2$; hence

$$\sum_1^n (u_i - \bar{u})^2 \simeq (n - 1)\sigma^2 = \frac{(n - 1)\bar{x}}{s^2} = \frac{(n - 1)\bar{u}}{s}$$

and the expectation value of Eq. (2.24) is

$$\chi^2 = \frac{s}{\bar{u}} \frac{(n - 1)\bar{u}}{s} = n - 1 \tag{2.25}$$

as in Eq. (3.1) of Chap. 27. The number of degrees of freedom would again be $F = n - 1$, and Fig. 2.1 of Chap. 27 is to be used with χ^2 from Eq. (2.24).

Case II, Constant Number of Counts. In a series of n successive observations, let the total time required to accumulate a predetermined number of input counts s be $t_1, t_2, \ldots, t_n$. Then the average interval for s input counts is

$$\bar{t} = \frac{1}{n} \sum_1^n t_i \tag{2.26}$$

Because of the general requirement that statistically distributed parameters be dimensionless, we cannot proceed toward χ^2 by forming

$$\sum \left[\frac{(t_i - \bar{t})^2}{\bar{t}} \right]$$

which would have dimensions of time and a numerical value which would depend on the units of time (seconds, minutes, etc.) used in the observations.

We may proceed by obtaining from our observations of t a substantially equivalent distribution of the effective number of input counts in a hypothetical and arbitrary fixed time interval T, which can be given the value $\bar{t}$ without loss of generality. Then if s input counts require a time t, the average rate for this interval is s/t and the corresponding number of input events x, in a fixed time $\bar{t}$, would be expected to be approximately

$$x \simeq \frac{s}{t} \bar{t} \tag{2.27}$$

while

$$\bar{x} = s \tag{2.28}$$

Then we can form a corresponding dimensionless expression for χ^2, by the substitutions

$$\chi^2 = \sum_1^n \frac{(x_i - \bar{x})^2}{\bar{x}} = \sum_1^n \frac{[(s\bar{t}/t_i) - s]^2}{s}$$

$$= s \sum_1^n \left(\frac{\bar{t}}{t_i} - 1 \right)^2 \tag{2.29}$$

The expected value of the summation in Eq. (2.29) is

$$\sum_{1}^{n} \left(\frac{\bar{t}}{t_i} - 1\right)^2 = (n - 1) \int_0^\infty \left(\frac{\bar{t}}{t} - 1\right)^2 dP_t \tag{2.30}$$

Inserting dP_t from Eq. (2.3) and $\bar{t} = s/a$ from Eq. (2.9), expanding, and evaluating the three resulting integrals lead to

$$(n - 1) \int_0^\infty \left(\frac{\bar{t}}{t} - 1\right)^2 dP_t = (n - 1) \frac{s + 2}{(s - 1)(s - 2)} \tag{2.31}$$

or approximately

$$(n - 1) \int_0^\infty \left(\frac{\bar{t}}{t} - 1\right)^2 dP_t \simeq \frac{n - 1}{s}\left(1 + \frac{5}{s} + \cdots\right) \simeq \frac{n - 1}{s} \tag{2.32}$$

for large values of s (4,096, etc.), such as are customarily used in scalers operating toward a preset number of counts. Then the expectation value of Eq. (2.29) is

$$\chi^2 = s\frac{n - 1}{s} = n - 1 \tag{2.33}$$

The number of degrees of freedom is again $F = n - 1$, and Fig. 2.1 of Chap. 27 is to be used with χ^2 from Eq. (2.29).

g. Effects of Resolving Time at Input and Output of Scaler. Counting losses due to the finite reaction time of an electromechanical register or other output device can be made negligibly small through the use of an adequately large scaling factor. With a properly designed counting system, the losses can be restricted mainly to the Geiger-Müller counter or scintillation counter preamplifier, which provides the input to the scaling circuit. Such losses remove the shortest intervals in the distribution of input pulses to the scaler, as discussed in Sec. 1. An exact reanalysis of subsequent losses in the various scaling stages would be both complicated and of little practical value. An upper limit to the losses in the scaler can be computed readily by assuming that the scaler input is truly random.

Then the fractional losses of s-fold pulses from the sth stage is given simply by Eq. (2.8) with T set equal to the resolving time ρ_s of the sth stage. For $a\rho_s \ll 1$, $P_{s+1} \ll P_s$, and Eq. (2.8) reduces simply to

$$P_T \simeq \frac{(a\rho_s)^s}{s!} \tag{2.34}$$

This is really an overestimate of the fractional loss in the sth stage because actually many of these pulses will have been lost in previous stages. In scalers of the 2^n variety, composed of cascaded scales of 2, only the resolving time of the early stages need be very short (D34). DeVault has developed a circuit in which $\rho_8 = 2\rho_4 = 8\rho_2 = 40\rho_1$, and he estimates that, if the first stage misses 1 per cent of the pulses, subsequent stages should not lose more than an additional 0.1 per cent.

If the input counting rate is large, and the scaling factor s is too small, the resolving time of the output register may become the governing factor in determining counting losses. This situation is more commonly encountered with scintillation counters than with Geiger-Müller counters because of the very short resolving time which can be realized with scintillation counters. If two s-fold output pulses occur within the resolving time ρ_s of the recorder, then the electronic scaling circuit clears and begins counting over at 0, 1, 2, . . . input pulses, without recording the second s-fold pulse. Electromechanical registers generally are a Type I (paralyzable) apparatus, as described in Sec. 1, and therefore respond only to intervals which are longer than their resolving time. The ratio of observed to true counting rate is, by analogy with Eq. (1.1), the fraction of s-fold intervals which are longer than ρ, or, from Eq. (2.6),

$$1 - P_\rho = e^{-a\rho} + a\rho e^{-a\rho} + \tfrac{1}{2}(a\rho)^2 e^{-a\rho} + \cdots + \frac{1}{(s-1)!}(a\rho)^{s-1}e^{-a\rho} \tag{2.35}$$

Problems

1. A β-ray counter has both a direct recording output and a scale-of-4 output. The following readings were made in successive 5-min intervals on the direct output (scale of 1): 200, 215, 195, 175, 225, 205, 185, 205, 190, 180, 210, 230.

(*a*) Compute the average value and its probable error as determined by the residuals. Compare with the probable error expected if the data follow a Poisson distribution.

(*b*) What will be the output of the scale of 4 for each of the 5-min intervals (remember to carry over 0, 1, 2, or 3 counts from each interval to the next interval)?

(*c*) Compute the average value of the scale-of-4 counts and its probable error from the residuals.

(*d*) Compute the probable error expected if the scale-of-4 data follow a Poisson distribution.

(*e*) Calling the solution to (*c*) $\bar{y} \pm r$, is the correct value for the average number of β rays per 5-min interval given by $4\bar{y} \pm 4r$?

(*f*) Calling the solution to (*d*) $\bar{y} \pm s$, is the correct value for the average given by $4\bar{y} \pm 4s$? Why? Compare with solution to (*a*).

2. In 22 successive 30-min intervals a scale-of-2 output gave the following numbers of pulses: 10, 15, 17, 9, 17, 15, 13, 16, 13, 14, 17, 21, 11, 12, 16, 8, 15, 8, 7, 15, 15, 8.

(*a*) Compute the average rate of the scale-of-2 process.

(*b*) Compare the standard error from the residuals with the expected S.E. based on the average number of scale-of-2 counts. Does the dispersion of the data seem excessive?

(*c*) Compute χ^2 and compare with the expected χ^2 if the scale-of-1 process is random.

(*d*) Estimate the probability that a truly random process would give a larger dispersion.

3. Verify Eq. (2.6) by integration of Eq. (2.5).

4. Show that the standard deviation of the duration of s-fold intervals is $\sqrt{s}/a$, where a is the average rate of random input events.

5. Show that Eq. (2.29) can be expressed in the more convenient form

$$\chi^2 = \frac{s}{(\bar{t})^2} \sum_{1}^{n} (\bar{t} - t_i)^2$$

if $\bar{t}/t_i$ is nearly unity.

6. A certain scaling circuit has an output which operates a printing timer after $10 \times 4{,}096 = 40{,}960$ input pulses have been received. With a certain radioactive source of constant strength, the number of seconds required to accumulate 40,960 input counts in each of 15 separate runs was: 2,595; 2,616; 2,624; 2,632; 2,648; 2,610; 2,638; 2,597; 2,605; 2,619; 2,622; 2,626; 2,615; 2,618; 2,623. Do these data satisfy the χ^2 test for randomicity of the primary process? *Ans.:* $\chi^2 = 16.9$; $F = 14$; $P \simeq 0.2$; yes.

7. Random pulses from a scintillation counter are fed into a scale of 8 which actuates a mechanical register. Two output pulses from the scale of 8 within a time interval of 5×10^{-2} sec will not be resolved by the mechanical register; only the first pulse will be recorded. If the register is counting at the rate of 600 per minute, what is the true rate at which pulses are arriving at the input of the scale of 8? Neglect all counting losses besides those in the mechanical register. *Ans.:* 5,160 counts per minute.

3. *Counting-rate Meters*

In counting-rate meters (often called CRM) each pulse from a counter is converted electronically into a charge q which is added to the charge Q on a tank condenser C. A resistance R shunts the tank condenser. The charge Q on the condenser can be read continuously, either by reading the potential difference

$$V = \frac{Q}{C} \tag{3.1}$$

across the condenser with a vacuum-tube voltmeter, or by reading the current

$$i = \frac{V}{R} = \frac{Q}{RC} \tag{3.2}$$

through the shunt resistance.

The statistical interpretation of the counting-rate-meter output readings due to randomly distributed input pulses requires a special statistical theory (S12, K19) because the integrating and averaging circuit RC produces an *exponential interdependence of successive observations on all preceding observations.*

a. Average Rate. Let the charge Q on the tank condenser be zero at $t = 0$, when a radiation source begins producing randomly distributed input pulses at a constant average rate a. The average number of pulses during the time interval from t to $t + dt$ will be $a\,dt$, and the expected increment of charge on the condenser in this interval is $qa\,dt$. If now a reading of Q is taken at a later time t_0, this increment of charge will have decayed to $qae^{-(t_0-t)/RC}\,dt$, because of leakage of charge through the resistance, with the time constant RC of the tank circuit. Thus the *expectation value* Q_m for the charge at any time t_0 is

$$Q_m(t_0) = \int_0^{t_0} qae^{-(t_0-t)/RC}\,dt = qaRC(1 - e^{-t_0/RC}) \tag{3.3}$$

and the expected equilibrium charge, after the counting-rate meter has been operating for a time $t_0 \gg RC$, has the value

$$Q_m = qaRC \tag{3.4}$$

It will be noted that Q_m is simply the specific charge q per pulse times the average number of pulses aRC occurring in one time constant RC of the tank circuit. Alternatively, each pulse can be considered to have a mean life RC in the tank circuit. Then, by analogy with radioactive series decay, the mean number of pulses in the tank at equilibrium is the pulse rate a times the mean life RC; compare Eq. (3.27).

We note also that the mean potential difference across the tank is

$$V_m = \frac{Q_m}{C} = qaR \tag{3.5}$$

which is independent of C. This is equivalent to noting that at equilibrium Q is constant, and that therefore the average current qa simply passes through the resistance R, in which it produces a potential difference qaR, by Ohm's law.

b. Standard Deviation of a Single Reading. Suppose that the counting-rate meter has been operating for a time $t_0 \gg RC$, and we make a single observation of the charge Q on the tank condenser at time t_0. The mean expected value is Q_m, but individual single observations will be distributed about Q_m with some standard deviation $\sigma(Q)$, which we now must evaluate.

Because a is randomly distributed, the number of events in a small time interval obeys Poisson's distribution, and the standard deviation of the number of events $a\,dt$ between t and $t + dt$ is $(a\,dt)^{\frac{1}{2}}$. Then the standard deviation of the increment of charge is $q(a\,dt)^{\frac{1}{2}}$. When observed at a later time, t_0, this deviation will make a contribution $q(a\,dt)^{\frac{1}{2}}e^{-(t_0-t)/RC}$ to the deviation of Q at t_0. All such contributions are statistically independent. Therefore, their total effect is to be obtained from the sum of the squares of the individual deviations, by the usual principles of the propagation of errors. Hence the total variance $\sigma^2(Q)$, of Q at t_0, is

$$\begin{aligned}\sigma^2(Q) &= \int_0^{t_0} q^2ae^{-2(t_0-t)/RC}\,dt \\ &= \tfrac{1}{2}q^2aRC(1 - e^{-2t_0/RC})\end{aligned} \tag{3.6}$$

and the variance for single observations of Q when $t_0 \gg RC$ is

$$\sigma^2(Q) = \tfrac{1}{2}q^2aRC \tag{3.7}$$

Thus the variance is only one-half as great as would be given by the

Poisson variance aRC of the pulses received in one time constant RC. This is because the square-law dependence on individual fluctuations emphasizes the fluctuations which have occurred only a short time before the reading is taken at t_0.

The *fractional* standard deviation of a single instantaneous reading of the counting-rate meter is then

$$\frac{\sigma(Q)}{Q_m} = \frac{1}{\sqrt{2aRC}} \tag{3.8}$$

or the same as would be expected in a direct counting observation over a time $2RC$.

c. Standard Deviation of the Average of n Independent Readings. Suppose that at $t_0 \gg RC$ a single reading of Q_1 is taken, and that additional readings are taken at subsequent times ϑRC, $2\vartheta RC$, . . . , $(n - 1)\vartheta RC$. Then, because the expectation value of Q is the same for each reading, the best estimate of the true mean value of Q will be given by the arithmetic average of these n readings

$$Q_m \simeq \bar{Q} \equiv \frac{1}{n} \sum_{1}^{n} Q_i \tag{3.9}$$

The standard error of this average value $\bar{Q}$ about the true mean value Q_m will be given by the usual principles [Chap. 26, Eq. (2.12)] only if $\vartheta \gg 1$. In the more applicable cases of small ϑ, the successive readings of Q are not statistically independent of one another, because of the exponential memory of the tank circuit. We may consider as our set of truly independent readings those parts of the various readings which are due to charge accumulated *since* the preceding reading. Only this much of any reading is independent of preceding readings. We must now determine the manner in which the standard error of the mean value $\bar{Q}$ depends on both n and ϑ.

The second reading Q_2, which is taken at a time ϑRC after Q_1, will have the value

$$Q_2 = Q_1 e^{-\vartheta} + G_1 \tag{3.10}$$

Here $Q_1 e^{-\vartheta}$ is the residual charge from the decay of Q_1, and G_1 is due to new charge accumulated between t_0 and $t_0 + \vartheta RC$. Because all the readings Q_i have the same expectation value Q_m, the expectation value $E[G_i]$ of G_i can be written from Eq. (3.10) or (3.3) and is

$$E[G_i] = (1 - e^{-\vartheta})Q_m \tag{3.11}$$

For algebraic convenience we will hereafter use the definition

$$r \equiv e^{-\vartheta} \tag{3.12}$$

where r denotes the residual fraction of any observation Q_i which is present in the subsequent observation Q_{i+1}, by Eq. (3.10).

The series of n consecutive observations Q_1, Q_2, . . . , Q_n, equally spaced in time, are exponentially linked by the following recurrence relations

$$
\begin{aligned}
Q_2 &= rQ_1 + G_1 \\
Q_3 &= rQ_2 + G_2 = r^2Q_1 + rG_1 + G_2 \\
&\cdots\cdots\cdots\cdots\cdots \\
Q_i &= rQ_{i-1} + G_{i-1} \qquad (1 < i \leq n) \\
&= r^{i-1}Q_1 + r^{i-2}G_1 + \cdots + rG_{i-2} + G_{i-1}
\end{aligned}
\tag{3.13}
$$

The mean value $\bar{Q}$ of Eq. (3.9) is obtained by collecting terms in Eq. (3.13) and has the value

$$
\bar{Q} = \frac{1}{n}[(1 + r + \cdots + r^{n-1})Q_1 + (1 + r + \cdots + r^{n-2})G_1 + \cdots + (1 + r + \cdots + r^{n-i-1})G_i + \cdots + G_{n-1}] \tag{3.14}
$$

Summing the indicated geometric series, we obtain the simpler expression

$$
n\bar{Q} = \frac{1}{1-r}[(1 - r^n)Q_1 + (1 - r^{n-1})G_1 + \cdots + (1 - r^{n-i})G_i + \cdots + (1 - r)G_{n-1}] \tag{3.15}
$$

The quantities $Q_1, G_1, G_2, \cdots, G_{n-1}$ are all independent. Therefore the variance in $n\bar{Q}$ is the sum of the variances of each of the terms in Eq. (3.15). The expectation value of the variance of any Q_i is given by Eq. (3.7) and is the same for all readings. The expectation value for the variance $\sigma^2(G)$ of any G_i can then be obtained with the help of the recurrence relations, Eqs. (3.10) and (3.13). From the principles of the propagation of errors, we can write the expectation values for the variances as

$$
\sigma^2(Q) = \sigma^2(rQ) + \sigma^2(G) \qquad \text{for any } i \tag{3.16}
$$

The residue of any $\sigma(Q)$ at a later time ϑRC is $r\sigma(Q)$. Therefore Eq. (3.16) becomes

$$
\sigma^2(Q) = [r\sigma(Q)]^2 + \sigma^2(G) \tag{3.17}
$$

Therefore the expectation value of the variance $\sigma^2(G)$ for any G_i is given by

$$
\sigma^2(G) = (1 - r^2)\sigma^2(Q) = (1 - r^2)\tfrac{1}{2}q^2aRC \tag{3.18}
$$

which is confirmed by Eq. (3.6). By applying the principles of the propagation of errors to Eq. (3.15), and making use of Eq. (3.18), we can write for the variance of $n\bar{Q}$

$$
\sigma^2(n\bar{Q}) = \frac{\sigma^2(Q)}{(1-r)^2}[(1 - r^n)^2 + (1 - r^{n-1})^2(1 - r^2) + \cdots + (1 - r^{n-i})^2(1 - r^2) + \cdots + (1 - r)^2(1 - r^2)] \tag{3.19}
$$

When the indicated summations are carried out we obtain

$$
\sigma^2(n\bar{Q}) = \frac{\sigma^2(Q)}{(1-r)^2}[n(1 - r^2) - 2r(1 - r^n)] \tag{3.20}
$$

Then the standard deviation, or standard error, in $\bar{Q}$ can be written as

$\sigma_n(\bar{Q})$ and from Eqs. (3.20) and (3.12) has the value

$$\sigma_n(\bar{Q}) = \frac{1}{n}\sigma(n\bar{Q})$$
$$= \frac{1}{n(1-e^{-\vartheta})}[n(1-e^{-2\vartheta}) - 2e^{-\vartheta}(1-e^{-n\vartheta})]^{\frac{1}{2}}\sigma(Q) \quad (3.21)$$

Equation (3.21) is the relationship sought for the standard error in the average value $\bar{Q}$ of Eq. (3.9). If $\vartheta \gg 1$, so that the n readings are really statistically independent, then Eq. (3.21) correctly reduces to

$$\sigma_n(\bar{Q}) = \frac{\sigma(Q)}{\sqrt{n}} \quad \text{for } \vartheta \gg 1 \quad (3.22)$$

which is in accord with the elementary principles given by Eq. (2.12) of Chap. 26.

d. Standard Deviation of Continuous Observations. The output of a counting-rate meter is often a recording voltmeter or galvanometer. The continuous observation of Q over a finite time T corresponds to an infinite number of single readings, with an infinitely close spacing, or $n \to \infty$, $\vartheta \to 0$, but such that

$$n\vartheta = \frac{T}{RC} \quad (3.23)$$

Then the standard error $\sigma(T)$ of the average deflection $\bar{Q}$ is obtained by substituting Eq. (3.23) into Eq. (3.21). In the limit of $\vartheta \ll 1$, this leads to†

$$\frac{\sigma(T)}{\sigma(Q)} = \left\{\frac{2RC}{T}\left[1 - \frac{RC}{T}\left(1 - e^{-T/RC}\right)\right]\right\}^{\frac{1}{2}} \quad (3.24)$$

where $\sigma(Q) = q(aRC/2)^{\frac{1}{2}}$ is the standard deviation of a single instantaneous observation as given by Eq. (3.7). Figure 3.1 is a plot of the dependence of $\sigma(T)/\sigma(Q)$ on T/RC as given by Eq. (3.24).

When $T \gg RC$, the general expression of Eq. (3.24) reduces to

$$\frac{\sigma(T)}{\bar{Q}} \simeq \frac{\sigma(T)}{Q_\infty} = \frac{1}{\sqrt{aT}} \quad (3.25)$$

which is equivalent to Eq. (2.15) of Chap. 26 because aT is the total number of pulses observed in the interval T.

In the practical uses of a continuously recorded output, the mean deflection $\bar{Q}$ divides the instantaneous readings into two equal areas, as illustrated in Fig. 3.2. The fractional standard deviation of a single

† Equations (3.21) and (3.24) are rigorous and were first obtained by R. E. Burgess, *Rev. Sci. Instr.*, **20**: 964L (1949). They replace equations developed in 1936 by Schiff and Evans (S12) which are algebraically dissimilar but which give substantially identical numerical values, as in Fig. 3.1. The derivation of Eq. (3.21) by means of the recurrence relationships of Eq. (3.13) is due to Professor George P. Wadsworth and Dr. Joseph G. Bryan.

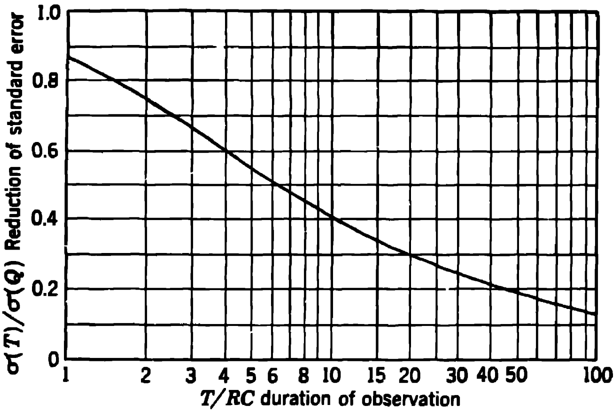


Fig. 3.1 Dependence of the ratio of the standard error $\sigma(T)$ of $\bar{Q}$ to the standard deviation $\sigma(Q)$ of a single observation, for continuous observations of various duration T/RC, Eq. (3.24).

observation can also be estimated directly from the recorded output, because the readings will exceed twice the standard deviation of a single observation only 4.6 per cent of the time. Thus if we draw dotted lines, as in Fig. 3.2, which include all but about 2 per cent of the highest observations and 2 per cent of the lowest observations, we shall have made a reasonably accurate graphical evaluation of the standard deviation $\sigma(Q)$ of any single point on the curve. From this, the standard error of the mean value can be obtained from Eq. (3.24), which is plotted in Fig. 3.1.

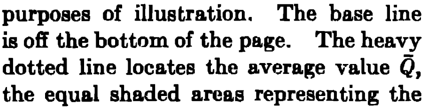


Fig. 3.2 Schematic representation of counting-rate-meter output, displaying exaggerated statistical fluctuations for purposes of illustration. The base line is off the bottom of the page. The heavy dotted line locates the average value $\bar{Q}$, the equal shaded areas representing the observations above and below the average value. The light dotted lines locate the region $\bar{Q} \pm 2\sigma(Q)$, where $\sigma(Q)$ is the standard deviation of any single point on the curve. The standard error of the average $\bar{Q}$ is then obtained with $\sigma(Q)$ and Fig. 3.1.

e. Equilibrium Time. Equation (3.3) shows that the approach to an equilibrium output is a characteristic exponential growth curve, similar in every respect to the growth of activity in a radioactive daughter substance, of mean life RC, from a long-lived parent radioactive substance of constant activity. For practical measurements, we may say that a condition experimentally indistinguishable from equilibrium exists when the charge $Q(t_0)$ differs from the average value by less than one probable error [$= 0.6745\sigma(Q)$] occasioned by statistical fluctuations. Then the time t_0 necessary to establish this practical equilibrium is

$$Q(t_0)e^{-t_0/RC} = \frac{0.6745Q(t_0)}{\sqrt{2aRC}}$$

from which
$$t_0 = RC(0.394 + \tfrac{1}{2}\ln 2aRC) \tag{3.26}$$

Figure 3.3 shows this dependence of t_0 on the counting rate a for several common values of RC.

f. Use of Counting-rate Meter on Rapidly Decaying Sources. It can be shown (S12) that the mean output of a counting-rate meter is always related to variations in the input in exactly the same way as the radioactivity of a daughter radioactive substance of mean life RC is related to the activity of its parent radioactive substance. The counting rate a is equivalent to the activity of the hypothetical parent radioactive substance. The term aRC is then equivalent to the number of atoms of daughter substance present at equilibrium, because it is of the form (activity × mean life). The exponential term $(1 - e^{-t_0/RC})$ in Eq. (3.3) is equivalent to the growth of a daughter substance of decay constant $(1/RC)$ from a long-lived parent of essentially constant activity. If the radiation source had been a radioactive substance of mean life τ, the input counting rate at any time t_0 would have been $ae^{-t_0/\tau}$ instead of the constant value a. Then the expectation value for the charge $Q_m(t_0)$ on the tank condenser at time t_0 would become

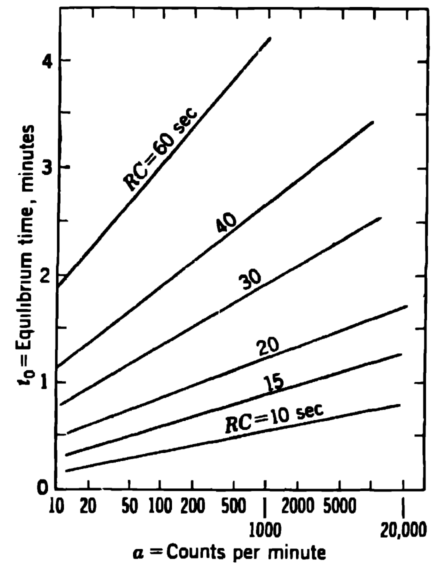


Fig. 3.3 Time t_0 required for the output to rise from zero to within one probable error of the final equilibrium value, for various counting rates a and time constants RC of the counting-rate meter.

$$Q_m(t_0) = qaRC \frac{1/RC}{(1/RC) - (1/\tau)} (e^{-t_0/\tau} - e^{-t_0/RC}) \tag{3.27}$$

which is entirely analogous to the amount of daughter substance present with a parent substance of decay constant $1/\tau$, as determined in Chap. 15. Similarly, the mean charge $Q_m(t_0)$ would pass through a maximum value at a time given by $[\ln (\tau/RC)]/[(1/RC) - (1/\tau)]$. After a time which is large compared with $[(1/RC) - (1/\tau)]$ the mean charge will be in transient equilibrium with the exponentially decreasing input counting rate.

Problems

1. Randomly distributed pulses, at an average rate a, are fed through a scale-of-s scaler and then into a counting-rate meter. Each scale-of-s pulse puts a charge q into the counting-rate-meter tank circuit whose time constant is RC.

(*a*) What is the equilibrium value of the tank-circuit voltage?

(*b*) Derive an analytical expression for the F.S.D. (fractional standard deviation) of a single observation of the equilibrium tank-circuit voltage.

(*c*) How does the expression obtained in (*b*) compare with the F.S.D if the scaler is removed from the circuit?

2. A Geiger-Müller counter is connected to a counting-rate meter having a tank circuit whose voltage is measured by a vacuum-tube voltmeter. The tank condenser has a capacitance of 10 μf and the specific charge per pulse is 10^{-10} coulomb. A long-lived radioactive source produces pulses in the Geiger-Müller tube at an average rate of 1,000 per minute. It is desired to have a 6-mv input to the voltmeter.

(*a*) What value should the tank resistor have?

(*b*) If a single reading of the voltmeter is taken after equilibrium has been established, what is its fractional standard deviation?

(*c*) If the output is recorded on a recording milliammeter for a 2-min period, what is the fractional standard error of the average output?

(*d*) If it is desired to read the equilibrium voltage as soon after the source is presented to the counter as possible, how long should one wait?

3. Compare the fractional standard deviation (F.S.D) of observations on a process whose random counting rate is 3,000 counts per minute, using a scale-of-100 scaler and alternatively a counting-rate meter whose time constant is $RC = 15$ sec.

(*a*) If the background counting rate can be neglected, how many seconds must one wait for the counting-rate meter to come to equilibrium?

(*b*) If a single observation is taken after equilibrium is reached, what is its F.S.D?

(*c*) If the counting-rate-meter readings for the next 75 sec are averaged, what is the F.S.D in the average value?

(*d*) If a single scaler reading is taken over the same 75 sec, what is its F.S.D?

(*e*) If a single scaler reading is taken over the same total period that the counting-rate meter is operating (equilibrium time + 75 sec), what is its F.S.D?

(*f*) Compare the numerical values in (*c*), (*d*), and (*e*) and explain clearly the statistical origin of the differences, especially why (*c*) is less than (*d*).

4. One minute is available for a measurement of the background rate of a certain discharge counter. A scaler and a counting-rate meter ($RC = 15$ sec) are available, and it is known that the background rate of this counter should be about 100 counts per minute.

(*a*) What will be the expected value of the F.S.D of the scaler reading?

(*b*) If the tank condenser is charged up to its equilibrium value in a few seconds by an adjustment of the calibration switch, what is the F.S.D of the counting-rate-meter reading?

(*c*) Explain any difference between the answers to (*a*) and (*b*).

(*d*) If the measurement time is now increased to 5 min, calculate the F.S.D for the scaler.

(*e*) What is the F.S.D of the counting-rate-meter reading, assuming that the calibration switch has been used to charge up the condenser quickly?

(*f*) What is the F.S.D of the counting-rate-meter reading, assuming that no "trick" is used to charge the condenser quickly?

(*g*) Explain any differences among the answers to (*d*), (*e*), and (*f*).

4. *Ionization Chambers*

The major features of the statistical behavior of ionization chambers can be inferred by comparison with the detailed statistical theory for counters. When ionization measurements are made by the *rate-of-drift* method, the total deflections are statistically analogous to readings of a total number of counts observed with a scaling circuit. When ionization measurements are made by the *steady-deflection* method, the readings are analogous to observations of a counting rate using a counting-rate meter. The controlling time constant RC is usually found in the input capacitance C and shunt resistance R of the ionization chamber and its electrometer circuit. Less frequently, the period of an output galvanometer may be the controlling time constant.

The theory of counter circuits thus leads toward an understanding of ionization-chamber circuits. However, one major characteristic of ionization chambers introduces mathematical complications which prevent the development of a detailed theory, except for unrealistic or trivial special cases.

In the statistical theory of counter circuits, each recorded ionizing particle produces the same effect on the instrument, namely, one count.

However, the number of ion pairs produced in an ionization chamber, per ionizing particle, not only depends on the type of particle but is statistically distributed about some mean value, even for identical particles of identical initial energy. Even an initially homogeneous group of α particles produces slightly different amounts of ionization, because of straggling. β rays, in addition to individual straggling, have a continuous initial distribution of energies and produce widely varying amounts of ionization per particle. Similar variations exist for the ionization per particle produced by the secondary electrons by which γ rays give rise to ionization, and by the ionizing recoil particles produced by fast neutrons.

Therefore, the statistical theory of counting-rate meters can be extended to ionization circuits only after realizing that the effectiveness q per particle is not constant but is distributed about some mean value. This distribution in q is generally not a random or Poisson one, nor even a normal distribution. It may be highly asymmetric, as is a primary spectrum of β-ray energies. Even its standard deviation may not be predictable on purely theoretical grounds. These fluctuations in q, that is, in the ionization per particle, therefore have the effect of increasing the observed fluctuations to some value greater than that predicted by the counter theory, in which q is a constant. Thus the statistical theory for the counting-rate meter gives the *lower limit* for the statistical fluctuations in an ionization current.

Similarly, the theory of scaling circuits serves as the *lower limit* for the statistical fluctuations in total ionization collected over a measured period of time, as in rate-of-drift measurements with an ionization chamber. In the theory of scaling circuits, the effectiveness per particle is implicitly taken as unity and is included in the term a of Eq. (2.1) et seq. for the average counting rate.

A second circumstance which can greatly increase the fluctuations in ionization is the fact that several types of ionizing particle may be acting simultaneously on the chamber. On this point, reference should be made to the generalized Poisson distribution in Chap. 26 and the illustrative examples given there. Circumstances can easily occur in which heavily ionizing particles, such as α rays or recoil protons, may produce only a small portion of the total ionization but may at the same time dominate the statistical fluctuations in ionization.

Finally, it should be pointed out that, when ionization chambers are operated as *proportional counters*, there are statistical fluctuations in the gas-amplification ratio. This again has the effect of making the specific effectiveness per particle [as represented by $a, b, \ldots$, in Eq. (3.1) of Chap. 26] a statistically distributed quantity. The general effect is to impair the resolution in studies of spectral distribution using proportional counters (H16). Similar statistical considerations apply to other multiplicative processes, such as the luminescent counter systems and photomultipliers used with fluorescent counters (S28).

5. *Rapid Decay of a Single Radionuclide*

In the derivation of the Poisson distribution we imposed the condition that the average rate of the process be constant over the period of the observations. Poisson's distribution and the statistical results of its simple application therefore cannot be applied directly to data which are taken in time intervals which are comparable with the mean life of the radioactive substance being studied. Poisson's distribution can be used as a basis for developing special statistical treatments which are applicable to measurements on rapidly decaying sources.

a. Mean-life Determination by Peierls's Method. The statistics of the rapidly decaying source have been treated in detail by Peierls (P14) whose minimum-error method for determining exponential coefficients is of general importance in many physical processes. For example, it applies to the determination of γ-ray absorption coefficients as well as to the determination of radioactive decay constants.

Peierls's method recognizes that the detecting apparatus possesses a finite background counting rate whose average rate is constant. Therefore the rapidly decaying radioactive source must be counted as an additional effect superimposed upon a statistically distributed background.

Peierls has shown that the radioactive decay constant λ is to be obtained with minimum error by designing the experiment so that the data can be used to calculate the *mean life* for the atoms whose disintegrations are observed. Figure 5.1 illustrates the method and the principal statistical problems which are encountered.

At time $t = 0$, let there be N_0 atoms whose disintegrations can be detected. (The actual initial number of atoms will be greater than N_0, except when a 4π-geometry detector possessing 100 per cent counting efficiency is used.) Then the initial instantaneous counting rate due to the source has the expectation value

$$\left(\frac{dN}{dt}\right)_{t=0} = N_0\lambda = \frac{N_0}{\tau} \tag{5.1}$$

where τ is the mean life of all the N_0 atoms. This initial counting rate should be at least several times the background counting rate.

Based upon preliminary approximate knowledge of the half-period and the initial activity, select a time interval Δt which is less than about one-third the mean life τ, yet is long enough to contain a statistically significant number of counts. In an unbroken sequence of n contiguous equal time intervals Δt, observe the total number of counts $x_1, x_2, x_3, \ldots, x_n$ due to the source plus the background. Then our best estimate of the number of counts $\Delta N_1, \Delta N_2, \ldots$, due to the source alone, is

$$\begin{aligned} \Delta N_1 &= x_1 - b\Delta t \\ \Delta N_2 &= x_2 - b\Delta t \\ &\cdots\cdots\cdots \\ \Delta N_n &= x_n - b\Delta t \end{aligned} \tag{5.2}$$

where $b\Delta t$ is the expectation value of the number of background counts in a time Δt.

Average Life of the Observed Atoms. The ΔN_1 atoms which decay during the first time interval Δt can be assigned an average lifetime of $\frac{1}{2}\Delta t$. Similarly, the ΔN_2 atoms which decay during the second time interval Δt to $2\Delta t$ can be assigned an average lifetime of $\frac{3}{2}\Delta t$. Then the total lifetimes of all the atoms which decay between $t = 0$ and $t = n\Delta t$ are

$$\Delta N_1(\tfrac{1}{2}\Delta t) + \Delta N_2(\tfrac{3}{2}\Delta t) + \Delta N_3(\tfrac{5}{2}\Delta t) + \cdots + \Delta N_n \frac{2n-1}{2}\Delta t \tag{5.3}$$

During this time, the total number of atoms whose individual lifetimes are observed is

$$N \equiv \Delta N_1 + \Delta N_2 + \Delta N_3 + \cdots + \Delta N_n \tag{5.4}$$

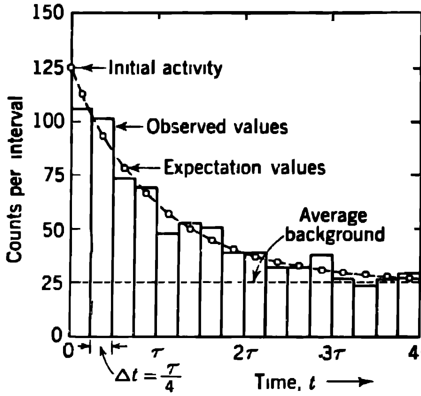


Fig. 5.1 Graphical presentation of typical statistical fluctuations encountered in measuring the mean life of a rapidly decaying source. The instantaneous value of the initial activity N_0/τ of this source is only four times the background. In a sequence of contiguous time intervals Δt, the expectation values of the number of counts per interval are shown as circles, plotted at the mid-points of the time intervals, and connected by the dotted decay curve. The actual observations, shown by the histogram, involve statistical fluctuations due both to background and to the source. Table 5.1 shows that in the calculation of the average life, by means of Eq. (5.5), observations beyond $T_0 \simeq 2.8\tau$ should be excluded. Each time interval here is $\Delta t = \tau/4$; therefore only the first $n = 11$ time intervals should be included in Eq. (5.5). After $t = n\Delta t = 2.75\tau$, the statistical fluctuations in the background are comparable with the residual activity of the source.

Hence the average life s of all the observed atoms is given by Eq. (5.3) divided by Eq. (5.4) which is

$$s = \frac{\Delta N_1 + 3\Delta N_2 + 5\Delta N_3 + \cdots + (2n-1)\Delta N_n}{\Delta N_1 + \Delta N_2 + \Delta N_3 + \cdots + \Delta N_n}\,\frac{\Delta t}{2} \tag{5.5}$$

By terminating the observations at the finite time $t = n\Delta t$ we have excluded the longest-lived atoms. Therefore the average life s of the *observed* atoms is less than the true mean life τ of *all* the atoms. The expectation value of s is given by

$$s = \frac{\int_0^{n\Delta t} t\, dN}{\int_0^{n\Delta t} dN} = \frac{(N_0/\tau)\int_0^{n\Delta t} te^{-t/\tau}\, dt}{(N_0/\tau)\int_0^{n\Delta t} e^{-t/\tau}\, dt} \tag{5.6}$$

$$s = \tau\left[1 - \frac{n\Delta t/\tau}{(e^{n\Delta t/\tau} - 1)}\right] \tag{5.7}$$

Equation (5.6) furnishes the means of calculating τ from the experimentally determined average life s of the observed atoms. Figure 5.2 relates the observed quantities s and $n\Delta t$ to the true mean life τ. It is obtained from Eq. (5.7) by assuming a series of arbitrary values $n\Delta t/\tau$ and computing the corresponding values of τ/s and hence $n\Delta t/s$.

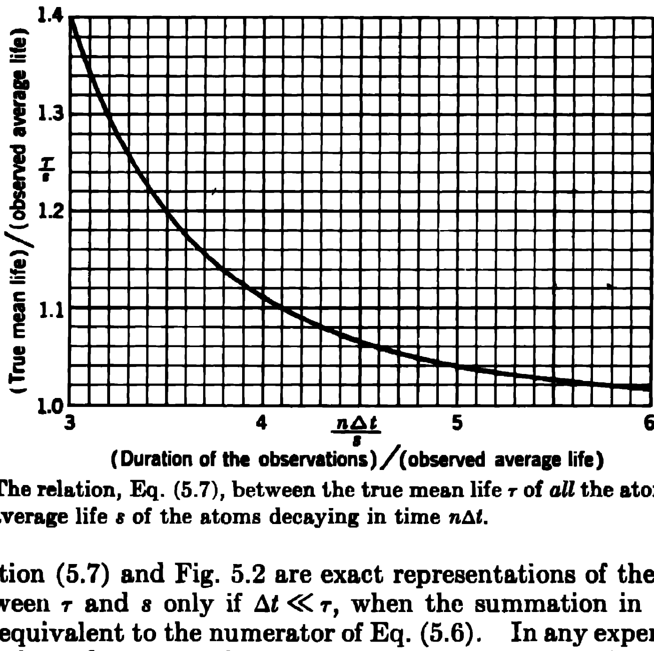


Fig. 5.2 The relation, Eq. (5.7), between the true mean life τ of *all* the atoms and the observed average life s of the atoms decaying in time $n\Delta t$.

Equation (5.7) and Fig. 5.2 are exact representations of the relationship between τ and s only if $\Delta t \ll \tau$, when the summation in Eq. (5.3) becomes equivalent to the numerator of Eq. (5.6). In any experiment, Δt must be finite and long enough to accumulate a significant number of counts. Peierls has shown that τ, as given by Eq. (5.7) and Fig. 5.2, is 1 to 1.5 per cent high if $\Delta t = 0.3\tau$. This systematic error in τ varies with $(\Delta t/\tau)^2$; therefore it is reduced to about 0.1 to 0.2 per cent if $\Delta t \leq 0.1\tau$. Actually, very few decay constants are known within an accuracy of 1 per cent.

Optimum Duration of Observations. If the measurements x_1, x_2, . . . are continued for many mean lives, the residual activity of the source may become small compared with the ever-present statistical fluctuations in the background. This situation can be seen in Fig. 5.1, beyond $t \sim 3\tau$. There comes a time when it is foolish to continue the measurements because the additional data which can be obtained are so inaccurate that their use will actually increase the error in the calculation of the mean life. Conversely, the measurements should not be discontinued too soon. Otherwise, data which would be useful statistically will not be obtained.

It is seen that there is some optimum time during which counting should be continued. This optimum duration T_0 is of the order of three to five mean lives for most practical cases. Peierls has shown that the minimum error in the determination of the mean life τ is obtained by selecting an optimum duration T_0 which depends upon the ratio of the initial activity of the source N_0/τ to the mean background rate b. These values are given in Table 5.1. The actual number of intervals used in Eq. (5.5) should therefore be chosen so that n is the nearest integer to $T_0/\Delta t$, that is,

$$n \simeq \frac{T_0}{\Delta t} \tag{5.8}$$

The initial activity N_0/τ of the source is, of course, greater than $\Delta N_1/\Delta t$. The preliminary estimates of the instantaneous initial activity and of the mean life, which are needed for the application of Peierls's method, are

TABLE 5.1. OPTIMUM DURATION T_0 OF OBSERVATIONS ON A RAPIDLY DECAYING SOURCE WHOSE MEAN LIFE IS τ, WHEN THE INITIAL RATIO OF SOURCE ACTIVITY TO BACKGROUND IS $N_0/\tau b$.

The table also gives typical values of the ratio τ/s of the true mean life τ to the average life s of the N atoms observed; the fraction N/N_0 of the atoms observed; and the standard error σ in the measurement of τ, as determined by Peierls (P14).

$\frac{N_0}{\tau}\frac{1}{b}$	2	5	10	30	65	100
$\frac{T_0}{\tau}$	2.6	3.0	3.5	4.0	4.5	5.0
$\frac{\tau}{s}$	1.263	...	1.122	1.081	...	1.035
$\frac{T_0}{s}$	3.29	.	3.93	4.32		5.18
$\frac{N}{N_0}$	0.925	.	0.970	0.982	. .	0.993
$\frac{\sigma}{\tau}\sqrt{N}$	2.84	.	1.81	1.47		1.23

obtained most readily from a semilogarithmic graph of the experimental values of $\Delta N/\Delta t$ [Eq. (5.2)] plotted at the mid-points of their respective time intervals.

Standard Error of Mean Life. One of the greatest advantages of Peierls's method is that it also allows a calculation of the standard error in τ whenever *counting* methods have been employed to obtain the data for Eqs. (5.2) and (5.5). Peierls's lengthy computations are summarized in Fig. 5.3, where the fractional standard error σ/τ is given as a function of the total number of counts N due to the source, Eq. (5.4), and of T_0/s. This standard error includes the effects of statistical fluctuations in the background, and of the decay of the source. In the limiting case of a negligible background, σ approaches the Poisson value

$$\sigma = \frac{\tau}{\sqrt{N_0}} \quad \text{for} \begin{cases} b \to 0 \\ T_0 \to \infty \\ N \to N_0 \end{cases} \tag{5.9}$$

Peierls's results have been confirmed by Bartlett (B17), using the powerful and elegant statistical method introduced by Fisher (F51) and known as the "method of maximum likelihood."

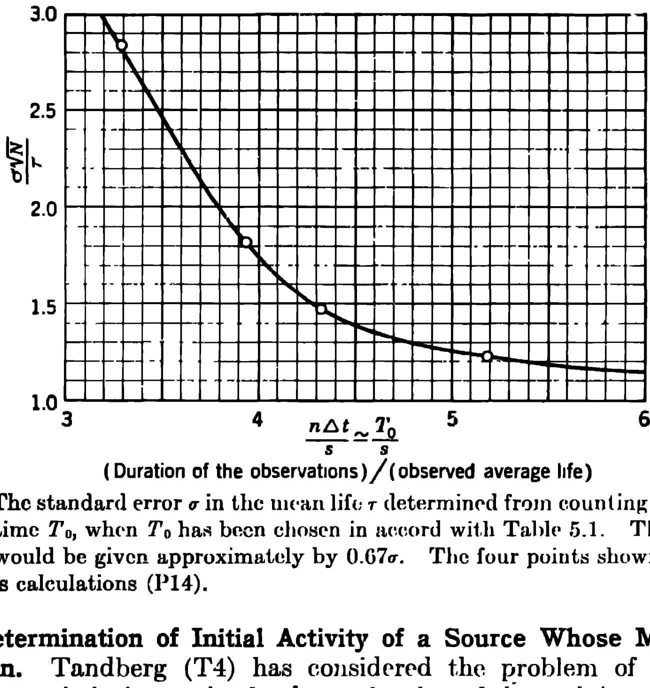


Fig. 5.3 The standard error σ in the mean life τ determined from counting N particles in a total time T_0, when T_0 has been chosen in accord with Table 5.1. The probable error in τ would be given approximately by 0.67σ. The four points shown are based on Peierls's calculations (P14).

b. Determination of Initial Activity of a Source Whose Mean Life Is Known. Tandberg (T4) has considered the problem of obtaining minimum statistical error in the determination of the activity of a radioactive source, from a single observation of the total number of counts observed in a time T_0. If the mean life τ of the source is known accurately, and the background counting rate b is comparable with the initial activity N_0/τ of the source, Tandberg showed that

$$\frac{N_0}{\tau}\frac{1}{b} = e^{T_0/\tau} - \frac{2T_0/\tau}{1 - e^{-T_0/\tau}} \tag{5.10}$$

where T_0 is the *optimum* duration of the single counting observation. In this interval T_0, the expected number of background counts is bT_0, the expected number of counts due to the source is $N_0(1 - e^{-T_0/\tau})$, and both counts are subject to random fluctuations. Values of T_0 for several values of $(N_0/\tau b)$, the ratio of initial source strength N_0/τ to background b, are given in the following table.

TABLE 5.2. OPTIMUM DURATION T_0 OF COUNTING WHEN τ IS KNOWN AND THE INITIAL ACTIVITY N_0/τ IS TO BE EVALUATED (T4)

$\frac{N_0}{\tau}\frac{1}{b}$	$\ll 1$	2	5	10	20
$\frac{T_0}{\tau}$	1 3	1.8	2.3	2.7	3.3

It will be noted that these values of the optimum counting time T_0 are slightly less than the values deduced by Peierls (Table 5.1) for the optimum duration of counting when τ is initially unknown and the best value of τ is to be determined from a *series* of observations on a single decaying source.

Problems

1. The usual experimental approximation to the true instantaneous activity dN/dt at any time t is obtained from the particle count ΔN over the finite time interval of duration Δt extending from $(t - \frac{1}{2}\Delta t)$ to $(t + \frac{1}{2}\Delta t)$. Show that this average activity $\Delta N/\Delta t$ always exceeds the instantaneous activity dN/dt at the mid-point of the time interval Δt and is given by

$$\frac{\Delta N}{\Delta t} = \frac{dN}{dt}\left[\frac{\sinh(\Delta t/2\tau)}{\Delta t/2\tau}\right] = \frac{dN}{dt}\left[1 + \frac{1}{24}\left(\frac{\Delta t}{\tau}\right)^2 + \cdots\right]$$

2. Show that the total lifetimes of the atoms which decay between $t = 0$ and $t = \Delta t$ are

$$\int_0^{\Delta t} t\,dN = \tfrac{1}{2}\Delta t\,\Delta N\left(1 - \frac{1}{6}\frac{\Delta t}{\tau} + \cdots\right)$$

3. Carry out the integrations indicated in Eq. (5.6) and show also that:

(*a*) The expectation value of N, the number of atoms whose disintegrations are observed, is

$$N = N_0(1 - e^{-n\Delta t/\tau})$$

(*b*) The expectation value of s becomes the ordinary mean life τ when the mean life is negligible compared with the duration of the observations.

4. Show that the greatest contribution to the sum of the lifetimes, Eq. (5.3), is due to those atoms which have an actual lifetime equal to the mean life τ.

5. Determine the half-period, and its standard error, from the following data. The average background of the counter is 25 counts per minute (scale of 1). In successive 1-min intervals, the number of counts due to source plus background is 106, 102, 73, 68, 48, 52, 51, 38, 38, 32, 32, 37, 27, 23, 26, 28. *Ans.*: From semilogarithmic graph of $\Delta N/\Delta t$: $N_0/\tau \simeq 100$ counts per minute; $\tau \simeq 4$ min; therefore, $T_0 \simeq 11$ min; $n = 11$. Finally, $\tau = 4.1 \pm 0.6$ (S.E.) min, or $T_{\frac{1}{2}} = 2.8 \pm 0.4$ (S.E.) min.

6. If the experimental conditions are such that the radionuclide studied in Prob. 5 cannot be made with greater initial activity, how many times must experiments like that of Prob. 5 be repeated in order to obtain a value of τ which has a standard error of 1 per cent?

7. Calculate by Peierls's method (*a*) the mean life and (*b*) the standard error in the mean life of a radioactive isotope, from the following data. Using a scale-of-2 recorder, the total number of impulses, including background, in successive 10-min intervals were 960, 820, 650, 560, 450, 420, 350, 330, 290, 260, 240, 230, 220, 210, 205. The scale-of-2 background is 180 per 10 min, the initial activity of the sample is about five times the background, and the half-period of the isotope is about 28 min.

6. *Radioactive-series Disintegrations*

In the derivation of the Poisson distribution we required that each event be independent of all others. In radioactive-series decay this condition is sometimes violated, depending on the time intervals chosen for observation. Thus AcA, the 0.0018-sec half-period decay product of Ac, gives α rays whose appearance is strongly governed by the decay of its parent product. The discovery of AcA by Geiger (G11) was directed by the excessive number of short intervals between successive α rays from actinon and its decay products, i.e., by deviations from the Poisson and interval distributions.

Cases of series decay can often be treated statistically by proper compounding of simple Poisson distributions. Adams (A8) has so treated the statistics of α-ray counting from Th in equilibrium with its decay products RdTh, ThX, Tn, ThA, ThB, ThC, etc. The half-period of ThX is 3.64 days. If in any 5-min observational interval a ThX α ray is counted, this will be closely followed by α rays from the successive disintegration products Tn and ThA (half-periods 54.5 sec and 0.158 sec, respectively). These latter α rays are therefore *not* randomly distributed in the time intervals chosen because they are dependent on the emission of the ThX α ray. Alternatively, it is seen that the total equilibrium amount of Tn and ThA present in this experiment is vanishingly small, being only those few atoms which have decayed out of the ThX state but not yet out of the ThA state. They therefore do not satisfy one of the conditions for the validity of Poisson's distribution because their probability of decay in the large time interval chosen is nearly unity, instead of nearly zero. Adams found the standard deviation for α counting in 5-min intervals on the Th series to be 1.45 times the Poisson value for a purely random process.

In general, the statistics of counting a series of substances which are in radioactive equilibrium can be deduced from the Poisson distribution if (S9) each radioactive mean life is either much longer or much shorter than the duration of the individual observations.