Form Factors and Charge Density Distributions

- $F(\vec{k}) = \int d^3r \; \rho(\vec{r}) \; e^{i\vec{k}\vec{r}}$
- $\rho(\vec{r}) = |\Psi(\vec{r})|^2 = \Psi^*(\vec{r})\Psi(\vec{r})$
- $\Psi(\vec{r}) = \frac{1}{\sqrt{2\pi}^3} \int d^3\tilde{p} \; e^{-i\tilde{\vec{p}}\vec{r}} \; \tilde{\Psi}(\tilde{\vec{p}}) = \mathcal{F}\tilde{\Psi}(\tilde{\vec{p}})$
- $\Psi(\vec{p}) = \frac{1}{\sqrt{2\pi}^3} \int d^3\tilde{r} \; e^{+i\vec{p}\tilde{\vec{r}}} \; \tilde{\Psi}(\tilde{\vec{r}}) = \mathcal{F}^{-1}\tilde{\Psi}(\tilde{\vec{r}})$

$$\Longrightarrow \quad F(\vec{k}) = \int d^3r \; \Psi^*(\vec{r})\Psi(\vec{r})e^{i\vec{k}\vec{r}}$$

$$= \int d^3r \; \Psi^*(\vec{r})\frac{1}{\sqrt{2\pi}^3} \int d^3\tilde{p} \; e^{-i(\tilde{\vec{p}}-\vec{k})\vec{r}} \; \tilde{\Psi}(\tilde{\vec{p}})$$

with $\vec{p} = \tilde{\vec{p}} - \vec{k} \Rightarrow d^3p = d^3\tilde{p}$

$$\Longrightarrow \quad F(\vec{k}) = \int d^3r \; \Psi^*(\vec{r})\frac{1}{\sqrt{2\pi}^3} \int d^3p \; e^{-i\vec{p}\vec{r}} \; \tilde{\Psi}(\vec{p}+\vec{k})$$

$$= \int d^3p \; \tilde{\Psi}(\vec{p}+\vec{k}) \left[\frac{1}{\sqrt{2\pi}^3} \int d^3r \; e^{+i\vec{p}\vec{r}} \; \Psi(\vec{r})\right]^*$$

$$= \int d^3p \; \tilde{\Psi}(\vec{p}+\vec{k})\tilde{\Psi}^*(\vec{p}) = F(\vec{k})$$

$$\Longrightarrow \quad \rho(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k \; F(\vec{k}) \; e^{-i\vec{k}\vec{r}}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \; \int d^3p \; \tilde{\Psi}(\vec{p}+\vec{k})\tilde{\Psi}^*(\vec{p}) \; e^{-i\vec{k}\vec{r}}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \; \int d^3p \; \tilde{\Psi}(\vec{p}+\vec{k}) \; e^{-i(\vec{p}+\vec{k})\vec{r}} \; \tilde{\Psi}^*(\vec{p}) \; e^{i\vec{p}\vec{r}}$$

with $\vec{p}+\vec{k} = \vec{p}\,' \Rightarrow d^3p' = d^3k$

$$\Longrightarrow \quad \rho(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3p' \; \int d^3p \; \tilde{\Psi}(\vec{p}\,') \; e^{-i\vec{p}\,'\vec{r}} \; \tilde{\Psi}^*(\vec{p}) \; e^{i\vec{p}\vec{r}}$$

$$= \Psi^*(\vec{r})\Psi(\vec{r}) = \rho(\vec{r})$$

Form Factors and Charge Density Distributions

- $F(\vec{k}) = \int d^3r\ \rho(\vec{r})\ e^{i\vec{k}\vec{r}}$
- $\rho(\vec{r}) = |\Psi(\vec{r})|^2 = \Psi^*(\vec{r})\Psi(\vec{r})$

$$\Longrightarrow \quad F(\vec{k}) = \int d^3r\ \rho(\vec{r})\ e^{i\vec{k}\vec{r}}$$
$$= \int d^3r\ \rho(\vec{r})\ \left(1 + i\vec{k}\vec{r} - \frac{(\vec{k}\vec{r})^2}{2} + \ldots\right)$$

if $\rho(\vec{r})$ spherically symmetric $\Rightarrow \rho(\vec{r}) = \rho(r)$

$$\Longrightarrow \quad F(k^2) = \iiint \left(1 + ikr\cos(\vartheta) - \frac{k^2r^2}{2}\cos^2(\vartheta) + \ldots\right) \rho(r) r^2 dr d\cos(\vartheta) d\varphi$$
$$= \iiint \rho(r) d^3r - \iiint \frac{k^2}{2}\cos^2(\vartheta) d\cos(\vartheta)\ \rho(r) r^4 dr\ d\varphi + \ldots$$

with $< r^2 >= \frac{\iiint r^2\rho(r)d^3r}{\iiint \rho(r)d^3r}$ and $\iiint \rho(r)d^3r = 1$

$$\Longrightarrow \quad F(k^2) = 1 - \frac{1}{6}k^2 < r^2 > + \ldots$$

$$\Longrightarrow \quad < r^2 > = -6\left(\frac{dF(k^2)}{dk^2}\right)_{k^2=0}$$